

“The Fermat Last Theorem”: a possible algebraic proof. Mathematical connections with string theory

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Abstract

In this paper, we have described in the **Section 1**, the Binomial Coefficient, in the **Section 2**, the Binomial Theorem and, in the **Section 3**, the Pitagora’s Theorem with algebraic method and with a trigonometric proof. In the **Section 4**, we describe an algebraic proof of the Fermat Last Theorem (FLT). In the **Section 5**, we have described some equations and theorems concerning the Wiles’s proof of FLT. In the **Sections 6, 7 and 8** we have described some equations concerning the p-adic, adelic strings and zeta strings. In conclusion, in the **Section 9**, we have describes some possible mathematical connections.

Introduction

There exist a simple way short and elegant to show the Fermat last Theorem, with methods of the epoch of Fermat among the '600 and the '700? We know that Newton and Fermat were contemporary.

were known different things:

- the interpolation of points with polynomials (method of the differences ended of Newton)
- the coefficients of Newton or the Tartaglia's Triangle or Pascal's triangle
- the diophantos equations
- methods of infinitesimal analysis
- etc

Fermat never showed the final solution. Was he deceived or didn't he want to show it? Did someone insist for seeing it? Fermat showed only a technique of the "descent infinite"... Eulero and others undertook complex paths. Andrew Wiles used the elliptic functions, tools of the 20° century. Did it serve all of this? Does a different path exist?

From when Wiles has shown the Theorem, a hunting is instigated to a simple solution, that involves professionals and amateurs. We have not subtracted there even, from a possible pindaric flight, or from a conceptual experiment.

It follows a possible demonstration, too much simple and, perhaps, for this elegant: are we deceived there or is it correct? If it is wrong, in thing it is wrong?

But before we describe some concepts that are at the base of the proof.

1. THE BINOMIAL COEFFICIENT

The binomial coefficient is defined from

$$C(n; k) = \binom{n}{k} = \frac{n!}{k! \cdot (n - k)!} \quad n, k \in \mathbb{Z}; 0 \leq k \leq n$$

(where $n!$ is the symbol of the factorial of n) and can be computed also by the Tartaglia's triangle.

It is useful in the calculation of the combinations $C(n;k)$ and it gives the number of the simple combinations of n elements of class k . For example:

$$\binom{5}{3} = \frac{5!}{3!(5-3)!} = \frac{120}{12} = 10$$

is the number of the combinations of 5 elements taken 3 to the time.

è il numero di combinazioni di 5 elementi presi 3 alla volta.

Property of the binomial coefficient

The binomial coefficient has the following properties:

- $\binom{n}{0} = \binom{n}{n} = 1$

Formal proof

$$\binom{n}{0} = \frac{n!}{0!(n-0)!} = \frac{n!}{n!} = 1$$

$$\binom{n}{n} = \frac{n!}{n!(n-n)!} = \frac{n!}{n!} = 1$$

Combinatorial proof

The combinations of n elements of length 0 or n are evidently only one: respectively the null set or the whole set of n elements

- $\binom{n}{1} = \binom{n}{n-1} = n$

Formal proof

$$\binom{n}{1} = \frac{n!}{1!(n-1)!} = \frac{n!}{(n-1)![n-(n-1)]!} = \binom{n}{n-1} = n$$

Combinatorial proof

There are evidently n ways to choose an element among n or to skip one of it

- $\binom{n}{k} = \binom{n}{n-k}$

Formal proof

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n!}{(n-k)![n-(n-k)]!} = \binom{n}{n-k}$$

Combinatorial proof

The choices of k elements are in biunivocal correspondence with the subsets of the n-k skipped elements

$$\bullet \binom{n+1}{k+1} = \binom{n}{k+1} + \binom{n}{k}$$

thence:

$$\bullet \binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$$

(This property allows to build the binomial coefficients with the Tartaglia's triangle).

Formal proof

$$\binom{n}{k+1} + \binom{n}{k} = \frac{n!}{(k+1)!(n-k-1)!} + \frac{n!}{k!(n-k)!}$$

We have that $(n-k)! = (n-k)(n-k-1)!$, and equivalently $(k+1)! = (k+1)k!$

we obtain:

$$\begin{aligned} \binom{n}{k+1} + \binom{n}{k} &= \frac{n!}{(k+1)k!(n-k-1)!} + \frac{n!}{(n-k)k!(n-k-1)!} = \\ &= \frac{(n-k)n!}{(k+1)(n-k)k!(n-k-1)!} + \frac{(k+1)n!}{(k+1)(n-k)k!(n-k-1)!} \end{aligned}$$

and thence:

$$\binom{n}{k+1} + \binom{n}{k} = \frac{(n-k+k+1)n!}{(k+1)k!(n-k)(n-k-1)!}$$

$$\binom{n}{k+1} + \binom{n}{k} = \frac{(n+1)!}{(k+1)!(n-k)!} = \binom{n+1}{k+1}$$

thence the thesis.

Combinatorial proof

To calculate the number of simple combinations of $n+1$ elements of length $k+1$, chooses one of the $n+1$ elements, that we will call X , and we divides the combinations in two classes: those that don't contain X and those that contain it. The cardinal properties of the two classes are evidently given by the two terms of the right-hand side of the formula that we wanted to show.

$$\bullet \quad 2^n = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \dots + \binom{n}{n-1} + \binom{n}{n}$$

Formal proof

From the binomial Theorem, we have that:

$$2^n = (1 + 1)^n = \sum_{k=0}^n \binom{n}{k} 1^{(n-k)} 1^k = \sum_{k=0}^n \binom{n}{k}$$

thence the thesis.

Combinatorial proof

2^n is the number of the subsets of a set of n elements. We can divide such subsets in classes, setting in every class those of a definite cardinality. Since the subsets of cardinality k are proper

$$\binom{n}{k},$$

we obtain the thesis immediately.

2. BINOMIAL THEOREM

The binomial theorem expresses the development of the n^{th} power of a binomial any with the following formula:

$$(a + b)^n = \sum_{k=0}^n \binom{n}{k} a^{n-k} b^k$$

where the factor:

$$\binom{n}{k}$$

represent the binomial coefficient and can be replaced with:

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

The formula is effective for every couple of real or complex numbers, but it is effective in general in every commutative ring.

How example of application of the formula, brings the cases $n = 2$, $n = 3$ and $n = 4$:

$$\begin{aligned}(x + y)^2 &= x^2 + 2xy + y^2 \\(x + y)^3 &= x^3 + 3x^2y + 3xy^2 + y^3 \\(x + y)^4 &= x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4.\end{aligned}$$

Proof

The binomial theorem can be shown for induction. Indeed it is possible to introduce for such theorem a basic step for which it trivially results true.

$$(a + b)^1 = \sum_{k=0}^1 \binom{1}{k} a^{(1-k)} b^k = a + b$$

It is possible to try with the inductive step, the truthfulness of the theorem for any exponent n . Indeed taken for corrected the following expression:

$$(a + b)^n = \sum_{k=0}^n \binom{n}{k} a^{(n-k)} b^k$$

that is true for $n+1$, one obtain

$$\begin{aligned}(a + b)^{n+1} &= (a + b)(a + b)^n \\&= (a + b) \sum_{k=0}^n \binom{n}{k} a^{n-k} b^k \\&= \sum_{k=0}^n \binom{n}{k} a^{n+1-k} b^k + \sum_{k=0}^n \binom{n}{k} a^{n-k} b^{k+1}\end{aligned}$$

and because is:

$$\sum_{k=0}^n \binom{n}{k} a^{n+1-k} b^k = \binom{n}{0} a^{n+1} + \sum_{k=1}^n \binom{n}{k} a^{n+1-k} b^k$$

$$\begin{aligned}
&= \binom{n}{0} a^{n+1} + \sum_{k=0}^{n-1} \binom{n}{k+1} a^{n+1-(k+1)} b^{k+1} \\
&= \binom{n}{0} a^{n+1} + \sum_{k=0}^{n-1} \binom{n}{k+1} a^{n-k} b^{k+1}
\end{aligned}$$

and

$$\sum_{k=0}^n \binom{n}{k} a^{n-k} b^{k+1} = \sum_{k=0}^{n-1} \binom{n}{k} a^{n-k} b^{k+1} + \binom{n}{n} b^{n+1}$$

we obtain that, using in the first passage a note property of the binomial coefficient:

$$\begin{aligned}
(a+b)^{n+1} &= \binom{n}{0} a^{n+1} + \sum_{k=0}^{n-1} \left(\binom{n}{k} + \binom{n}{k+1} \right) a^{n-k} b^{k+1} + \binom{n}{n} b^{n+1} \\
&= \binom{n}{0} a^{n+1} + \sum_{k=0}^{n-1} \binom{n+1}{k+1} a^{n-k} b^{k+1} + \binom{n}{n} b^{n+1} \\
&= \binom{n}{0} a^{n+1} + \sum_{k=1}^n \binom{n+1}{k} a^{n+1-k} b^k + \binom{n}{n} b^{n+1}
\end{aligned}$$

further, we know that:

$$\binom{n}{0} = \binom{n+1}{0} = 1 \quad \binom{n}{n} = \binom{n+1}{n+1} = 1$$

thence, we obtain:

$$\binom{n}{0} a^{n+1} + \sum_{k=1}^n \binom{n+1}{k} a^{n+1-k} b^k + \binom{n}{n} b^{n+1} = \binom{n+1}{0} a^{n+1} + \sum_{k=1}^n \binom{n+1}{k} a^{n+1-k} b^k + \binom{n+1}{n+1} b^{n+1}$$

and we obtain the formal expression of the development of the following power of the binomial

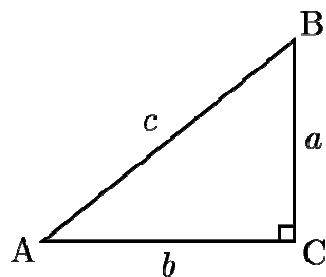
$$(a+b)^{n+1} = \sum_{k=0}^{n+1} \binom{n+1}{k} a^{(n+1)-k} b^k$$

that confirms the thesis.

3. PITAGORA's THEOREM

Proposition

In every right-angled triangle, the area of the square built on the hypotenuse is equivalent to the sum of the areas of the squares built on the catheti.



Gives a right-angled triangle of sides a, b and c, and pointing out with c the hypotenuse and with a and b the catheti, the theorem is express from the equation:

$$a^2 + b^2 = c^2$$

or, alternately, solving for c:

$$\sqrt{a^2 + b^2} = c.$$

and we obtain the respective catheti:

$$\sqrt{c^2 - b^2} = a.$$

and

$$\sqrt{c^2 - a^2} = b.$$

If the tern a, b, c is constituted by integer numbers it calls pitagoric tern.

Inversely, every triangle in which the three sides verify this property, is right-angled: this theorem, with his proof, appears in the Euclide's Elements immediately after the Pitagora's Theorem.

Trigonometric proof

The theorem of the sine connects the lengths of the sides of a triangle and the sine of the opposite angles. This relationship is also applied to any triangle and, in the case in which this is right-angled, can be equivalent to the Pitagora's Theorem (although in less immediate way in comparison to the theorem of the cosine).

The theorem of the sine affirms that in any triangle, the following relationships are true:

$$\frac{a}{\sin \alpha} = \frac{b}{\sin \beta} = \frac{c}{\sin \gamma} = 2R.$$

Raising to the square, one obtain:

$$c^2 = a^2 \frac{\sin^2 \gamma}{\sin^2 \alpha}, \quad b^2 = a^2 \frac{\sin^2 \beta}{\sin^2 \alpha}.$$

Adding the terms, we obtain:

$$c^2 + b^2 = a^2 \frac{\sin^2 \gamma + \sin^2 \beta}{\sin^2 \alpha}.$$

when α is a right angle, one obtain $\beta = \pi / 2 - \gamma$ and therefore:

$$\frac{\sin^2 \gamma + \sin^2 \beta}{\sin^2 \alpha} = \sin^2(\pi/2 - \beta) + \sin^2 \beta = \cos^2 \beta + \sin^2 \beta = 1.$$

Thence, we obtain the Pitagora's Theorem:

$$c^2 + b^2 = a^2.$$

4. Algebraic proof of the Fermat Last Theorem

Now we describe an our proof for n even and odd

Proof:

From the Pitagora's Theorem is:

$$x^2 + y^2 = c^2 \quad (1)$$

$$(x^2 + y^2)^m = (c^2)^m \quad (2)$$

From the binomial Theorem (Newton's formula) of (2) we have that:

$$\sum_{h=0}^m \binom{m}{h} (x^2)^{m-h} (y^2)^h = (c^2)^m$$

$$\binom{m}{0} (x^2)^m (y^2)^0 + \binom{m}{n} (x^2)^0 (y^2)^m + \sum_{h=1}^{m-1} \binom{m}{h} (x^2)^{m-1-h} (y^2)^h = (c^2)^m$$

$$x^{2m} + y^{2m} + \sum_{h=1}^{m-1} \binom{m}{h} (x^2)^{m-1-h} (y^2)^h = (c^2)^m \quad (3)$$

where

$$\binom{m}{0} = \binom{m}{m} = 1.$$

If $n = 2 \cdot m$, therefore n even, the eq. (3) become:

$$x^n + y^n + \sum_{h=1}^{m-1} \binom{m}{h} (x^2)^{m-1-h} (y^2)^h = c^n.$$

Because

$$\sum_{h=1}^{m-1} \binom{m}{h} (x^2)^{m-1-h} (y^2)^h > 0$$

thence, we have:

$$x^n + y^n < c^n.$$

If n is odd, then, for the following property of the square roots $\sqrt{n} + \sqrt{m} \geq \sqrt{n+m}$, we obtain:

$$x^n + y^n = (x^{2n})^{1/2} + (y^{2n})^{1/2} \geq (x^{2n} + y^{2n})^{1/2} > c^n$$

thence, for $n > 2$ is always

$$x^n + y^n \neq c^n.$$

Furthermore is:

$$\frac{x^n}{c^n} + \frac{y^n}{c^n} = 1$$

thence, we obtain:

$$a^n + b^n = 1 \quad (4)$$

where $a = \frac{x}{c}$ $b = \frac{y}{c}$

But the eq. (4) is an **elliptic curve** with rational and not integer solutions.

5. On some equations and theorems concerning the Wiles's proof of Last Fermat Theorem. [1]

Let w_f denote the number of roots of unity ζ of L such that $\zeta \equiv 1 \pmod{f}$ (f an integral ideal of O_L). We choose an f prime to p such that $w_f = 1$. Then there is a grossencharacter φ of L satisfying $\varphi((\alpha)) = \alpha$ for $\alpha \equiv 1 \pmod{f}$. According to Weil, after fixing an embedding $\overline{Q} \mapsto \overline{Q}_p$ we can associate a p -adic character φ_p to φ . We choose an embedding corresponding to a prime above p and then we find $\varphi_p = \kappa \cdot \chi$ for some χ of finite order and conductor prime to p .

The grossencharacter φ (or more precisely $\varphi \circ N_{F/L}$) is associated to a (unique) elliptic curve E defined over $F = L(f)$, the ray class field of conductor f , with complex multiplication by O_L and isomorphic over C to C/O_L . We may even fix a Weierstrass model of E over O_F which has good reduction at all primes above p . For each prime B of F above p we have a formal group \hat{E}_B , and this is a relative Lubin-Tate group with respect to F_B over L_p . We let $\lambda = \lambda_{\hat{E}_B}$ be the logarithm of this formal group.

Let U_∞ be the product of the principal local units at the primes above p of $L(fp^\infty)$; i.e.,

$$U_\infty = \prod_{B|p} U_{\infty,B} \quad \text{where} \quad U_{\infty,B} = \varprojlim U_n, B.$$

To an element $u = \varprojlim u_n \in U_\infty$ we can associate a power series $f_{u,B}(T) \in O_B[[T]]$ where O_B is the ring of integers of F_B . For B we will choose the prime above p corresponding to our chosen embedding $\overline{Q} \mapsto \overline{Q}_p$. This power series satisfies $u_{n,B} = (f_{u,B})(\omega_n)$ for all $n > 0, n \equiv 0 \pmod{d}$ where $d = [F_B : L_p]$ and $\{\omega_n\}$ is chosen as an inverse system of π^n division points of \hat{E}_B . We define a homomorphism $\delta_k : U_\infty \rightarrow O_B$ by

$$\delta_k(u) := \delta_{k,B}(u) = \left(\frac{1}{\lambda'_{\hat{E}_B}(T)} \frac{d}{dT} \right)^k \log f_{u,B}(T) \Big|_{T=0}. \quad (1)$$

Then

$$\delta_k(u^\tau) = \theta(\tau)^k \delta_k(u) \quad (2) \quad \text{for } \tau \in \text{Gal}(\bar{F}/F)$$

where θ denotes the action on $E[p^\infty]$. Now $\theta = \varphi_p$ on $\text{Gal}(\bar{F}/F)$. We want a homomorphism on u_∞ with a transformation property corresponding to ν on all of $\text{Gal}(\bar{L}/L)$. We observe that $\nu = \varphi_p^2$ on $\text{Gal}(\bar{F}/F)$.

Let S be a set of coset representatives for $\text{Gal}(\bar{L}/L)/\text{Gal}(\bar{L}/F)$ and define

$$\Phi_2(u) = \sum_{\sigma \in S} \nu^{-1}(\sigma) \delta_2(u^\sigma) \in \mathcal{O}_B[\nu]. \quad (3)$$

Each term is independent of the choice of coset representative by (17b) [1] and it is easily checked that

$$\Phi_2(u^\sigma) = \nu(\sigma) \Phi_2(u).$$

It takes integral values in $\mathcal{O}_B[\nu]$. Let $U_\infty(\nu)$ denote the product of the groups of local principal units at the primes above p of the field $L(\nu)$. Then Φ_2 factors through $U_\infty(\nu)$ and thus defines a continuous homomorphism

$$\Phi_2 : U_\infty(\nu) \rightarrow C_p.$$

Let C_∞ be the group of projective limits of elliptic units in $L(\nu)$. Then we have a crucial theorem of Rubin:

THEOREM 8.

There is an equality of characteristic ideals as $\Lambda = Z_p[[\text{Gal}(L(\nu)/L)]]$ -modules:

$$\text{char} \wedge (\text{Gal}(M_\infty/L(v))) = \text{char} \wedge (U_\infty(v)/\overline{C}_\infty).$$

Let $v_0 = v \bmod \lambda$. For any $Z_p[\text{Gal}(L(v_0)/L)]$ -module X we write $X^{(v_0)}$ for the maximal quotient of $X \otimes_{Z_p} \mathcal{O}$ on which the action of $\text{Gal}(L(v_0)/L)$ is via the Teichmüller lift of v_0 . Since $\text{Gal}(L(v)/L)$ decomposes into a direct product of a pro- p group and a group of order prime to p ,

$$\text{Gal}(L(v)/L) \cong \text{Gal}(L(v)/L(v_0)) \times \text{Gal}(L(v_0)/L),$$

we can also consider any $Z_p[[\text{Gal}(L(v)/L)]]$ -module also as a $Z_p[\text{Gal}(L(v_0)/L)]$ -module. In particular $X^{(v_0)}$ is a module over $Z_p[\text{Gal}(L(v_0)/L)]^{(v_0)} \cong \mathcal{O}$. Also $\Lambda^{(v_0)} \cong \mathcal{O}[[T]]$.

Now according to results of Iwasawa, $U_\infty(v)^{(v_0)}$ is a free $\Lambda^{(v_0)}$ -module of rank one. We extend Φ_2 \mathcal{O} -linearly to $U_\infty(v) \otimes_{Z_p} \mathcal{O}$ and it then factors through $U_\infty(v)^{(v_0)}$. Suppose that u is a generator of $U_\infty(v)^{(v_0)}$ and β an element of $\overline{C}_\infty^{(v_0)}$. Then $f(\gamma-1)u = \beta$ for some $f(T) \in \mathcal{O}[[T]]$ and γ a topological generator of $\text{Gal}(L(v)/L(v_0))$. Computing Φ_2 on both u and β gives

$$f(v(\gamma)-1) = \phi_2(\beta) / \Phi_2(u). \quad (4)$$

We have that v can be interpreted as the grossencharacter whose associated p -adic character, via the chosen embedding $\overline{Q} \mapsto \overline{Q}_p$, is v , and \bar{v} is the complex conjugate of v .

Furthermore, we can compute $\Phi_2(u)$ by choosing a special local unit and showing that $\Phi_2(u)$ is a p -adic unit.

Now, if we have that

$$\#H_{S_e}^1(Q_\Sigma/Q, Y) \leq \#(\mathcal{O}/\Omega^{-2}L_{f_0}(2, \bar{v})) \cdot \prod_{q \in \Sigma} \ell_q,$$

and

$$\#(\mathcal{O}/h_L) \cdot \prod_{q \in \Sigma - \{p\}} \ell_q, \quad (5)$$

where $\ell_q = \#H^0(Q_q, ((K/O)(\psi) \oplus K/O)^*)$ and h_L is the class number of O_L , combining these we obtain the following relation:

$$\#H_{Se}^1(Q_\Sigma/Q, V) \leq \#(O/\Omega^{-2}L_{f_0}(2, \bar{v})) \#(O/h_L) \cdot \prod_{q \in \Sigma} \ell_q, \quad (6)$$

where $\ell_q = \#H^0(Q_q, V^*)$ (for $q \neq p$), $\ell_p = \#H^0(Q_p, (Y^0)^*)$. (Also here, we remember that ℓ is p-adic).

Let ρ_0 be an irreducible representation as in (5) **[1]**. Suppose that f is a newform of weight 2 and level N , λ a prime of O_f above p and $\rho_{f, \lambda}$ a deformation of ρ_0 . Let m be the kernel of the homomorphism $T_1(N) \rightarrow O_f/\lambda$ arising from f .

We now give an explicit formula for η developed by Hida by interpreting \langle, \rangle in terms of the cup product pairing on the cohomology of $X_1(N)$, and then in terms of the Petersson inner product of f with itself. Let

$$(\cdot, \cdot): H^1(X_1(N), O_f) \times H^1(X_1(N), O_f) \rightarrow O_f \quad (7)$$

be the cup product pairing with O_f as coefficients. Let p_f be the minimal prime of $T_1(N) \otimes O_f$ associated to f , and let

$$L_f = H^1(X_1(N), O_f)[p_f].$$

If $f = \sum a_n q^n$ let $f^\rho = \sum \bar{a}_n q^n$. Then f^ρ is again a newform and we define L_{f^ρ} by replacing f by f^ρ in the definition of L_f . Then the pairing (\cdot, \cdot) induces another by restriction

$$(\cdot, \cdot): L_f \times L_{f^\rho} \rightarrow O_f. \quad (8)$$

Replacing O by the localization of O_f at p (if necessary) we can assume that L_f and L_{f^ρ} are free of rank 2 and direct summands as O_f -modules of the respective cohomology groups. Let δ_1, δ_2 be a basis of L_f .

Then also $\bar{\delta}_1, \bar{\delta}_2$ is a basis of $L_{f^\rho} = \bar{L}_f$. Here complex conjugation acts on $H^1(X_1(N), \mathcal{O}_f)$ via its action on \mathcal{O}_f . We can then verify that

$$(\delta, \bar{\delta}) := \det(\delta_i, \bar{\delta}_j)$$

is an element of \mathcal{O}_f whose image in $\mathcal{O}_{f,\lambda}$ is given by $\pi(\eta^2)$ (unit).

To give a more useful expression for $(\delta, \bar{\delta})$ we observe that f and \bar{f}^ρ can be viewed as elements of $H^1(X_1(N), \mathbb{C}) \cong H_{DR}^1(X_1(N), \mathbb{C})$ via $f \mapsto f(z)dz$, $\bar{f}^\rho \mapsto \bar{f}^\rho d\bar{z}$. Then $\{f, \bar{f}^\rho\}$ form a basis for $L_f \otimes_{\mathcal{O}_f} \mathbb{C}$. Similarly $\{\bar{f}, f^\rho\}$ form a basis for $L_{f^\rho} \otimes_{\mathcal{O}_f} \mathbb{C}$. Define the vectors $\omega_1 = (f, \bar{f}^\rho)$, $\omega_2 = (\bar{f}, f^\rho)$ and write $\omega_1 = C\delta$ and $\omega_2 = \bar{C}\bar{\delta}$ with $C \in M_2(\mathbb{C})$. Then writing $f_1 = f, f_2 = \bar{f}^\rho$ we set

$$(\omega, \bar{\omega}) := \det((f_i, \bar{f}_j)) = (\delta, \bar{\delta}) \det(C\bar{C}).$$

Now $(\omega, \bar{\omega})$ is given explicitly in terms of the (non-normalized) Petersson inner product $\langle \cdot, \cdot \rangle$:

$(\omega, \bar{\omega}) = -4\langle f, f \rangle^2$ where $\langle f, f \rangle = \int_{\mathfrak{S}/\Gamma_1(N)} f\bar{f} dx dy$. Hence, we have the following equation:

$$(\omega, \bar{\omega}) = -4 \left(\int_{\mathfrak{S}/\Gamma_1(N)} f\bar{f} dx dy \right)^2. \quad (9)$$

To compute $\det(C)$ we consider integrals over classes in $H_1(X_1(N), \mathcal{O}_f)$. By Poincaré duality there exist classes c_1, c_2 in $H_1(X_1(N), \mathcal{O}_f)$ such that $\det\left(\int_{c_j} \delta_i\right)$ is a unit in \mathcal{O}_f . Hence $\det C$ generates the same \mathcal{O}_f -module as is generated by $\left\{ \det\left(\int_{c_j} f_i\right) \right\}$ for all such choices of classes (c_1, c_2) and with $\{f_1, f_2\} = \{f, \bar{f}^\rho\}$. Letting u_f be a generator of the \mathcal{O}_f -module $\left\{ \det\left(\int_{c_j} f_i\right) \right\}$ we have the following formula of Hida:

$$\pi(\eta^2) = \langle f, f \rangle^2 / u_f \bar{u}_f \times (\text{unit in } \mathcal{O}_{f,\lambda}).$$

Now, we choose a (primitive) grossencharacter φ on L together with an embedding $\bar{\mathcal{Q}} \hookrightarrow \bar{\mathcal{Q}}_p$ corresponding to the prime p above \mathfrak{p} such that the induced p -adic character φ_p has the properties:

- (i) $\varphi_p \bmod \bar{\mathfrak{p}} = \kappa_0$ ($\bar{\mathfrak{p}}$ = maximal ideal of $\bar{\mathcal{Q}}_p$).
- (ii) φ_p factors through an abelian extension isomorphic to $Z_p \oplus T$ with T of finite order prime to p .
- (iii) $\varphi(\alpha) = \alpha$ for $\alpha \equiv 1(f)$ for some integral ideal f prime to p .

Let $p_0 = \ker \psi_f : T_1(N) \rightarrow \mathcal{O}_f$ and let $A_f = J_1(N) / p_0 J_1(N)$ be the abelian variety associated to f by Shimura. Over F^+ there is an isogeny $A_{f/F^+} \approx (E_{/F^+})^d$ where $d = [\mathcal{O}_f : \mathbb{Z}]$.

We have that the p -adic Galois representation associated to the Tate modules on each side are equivalent to $(\text{Ind}_F^{F^+} \varphi_0) \otimes_{\mathbb{Z}_p} K_{f,p}$ where $K_{f,p} = \mathcal{O}_f \otimes \mathbb{Q}_p$ and where $\varphi_p : \text{Gal}(\bar{F}/F) \rightarrow \mathbb{Z}_p^\times$ is the p -adic character associated to φ and restricted to F . We now give an expression for $\langle f_\varphi, f_\varphi \rangle$ in terms of the L -function of φ . We note that $L_N(2, \bar{\nu}) = L_N(2, \nu) = L_N(2, \varphi^2 \hat{\chi})$ and remember that ν is the p -adic character, and $\bar{\nu}$ is the complex conjugate of ν , we have that:

$$\langle f_\varphi, f_\varphi \rangle = \frac{1}{16\pi^3} N^2 \left\{ \prod_{\substack{q|N \\ q \in S_\varphi}} \left(1 - \frac{1}{q} \right) \right\} L_N(2, \varphi^2 \hat{\chi}) L_N(1, \psi), \quad (10)$$

where χ is the character of f_φ and $\hat{\chi}$ its restriction to L ; ψ is the quadratic character associated to L ; $L_N(\)$ denotes that the Euler factors for primes dividing N have been removed; S_φ is the set of primes $q|N$ such that $q = qq'$ with $q | \text{cond } \varphi$ and q, q' primes of L , not necessarily distinct.

6. On p-adic and adelic strings [2]

Open and closed p-adic strings.[2]

Let us now discuss the question of the construction of a dynamical theory for open and closed p-adic strings. It was proposed (Volovich, 1987) to consider p-adic generalization of the Veneziano string amplitude in two ways, according to two equivalent representations

$$A(a, b) = \int_0^1 |x|^{a-1} |1-x|^{b-1} dx = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}. \quad (1)$$

The first way corresponds to an interpretation of the amplitude $A(a, b)$ as a convolution of two characters and the second one to the p-adic interpolation of the gamma function. Using the first approach a complex-valued string amplitude over a finite Galois field has been constructed.

Consideration of string amplitudes as a convolution of characters is a very general concept applicable to characters on number fields, groups and algebras.

Now, we have the string amplitudes of the following form

$$A(\gamma_a, \gamma_b) = \int_K \gamma_a(x) \gamma_b(1-x) dx, \quad (2)$$

where K is a field F , i.e., $K = F$, $\gamma_a(x)$ is a multiplicative character on K , and dx is a measure on K .

Note that the range of integration in (2) is over the entire field F , and hence this p-adic generalization is rather one of the Virasoro-Shapiro amplitude

$$A = \int_C |z|^{a-1} |1-z|^{b-1} dz, \quad (3)$$

than of the Veneziano amplitude (1), where the integration is over the unit segment on the real axis. The equation (3) is just a particular case of (2) for $K = \mathbb{C}$ and $\gamma_a(z) = |z|^{a-1}$. The ordinary Veneziano amplitude can be rewritten in the following way

$$A = \int_R |x|^{a-1} |1-x|^{b-1} \theta_{[0,1]}(x) dx, \quad (4)$$

where $\theta(x)$ is the characteristic function of the segment $[0,1]$. In particular, it can be written in terms of the Heaviside function $\theta_{[0,1]}(x) = \theta(x)\theta(1-x)$. Hence, in order to have a generalization of the expression (4) on an arbitrary field F one should have on F an analogue of the Heaviside function or the function sign x .

We have a generalization of the amplitude (4), in the case of an arbitrary locally compact disconnected field F , in the following form

$$A_{F,\tau}^{open}(\gamma_a, \gamma_b) = \int_F |x|^{a-1} |1-x|^{b-1} \theta_{\tau[0,1]} dx \quad (5)$$

where $\theta_{\tau[0,1]}(x)$ is a p-adic generalization of the characteristic function of the segment $[0,1]$ on F related to a quadratic extension $F(\sqrt{\tau})$. In particular one can take the function $\theta_{\tau[0,1]}(x)$ in the form $\theta_{\tau}(x)\theta_{\tau}(1-x)$ where $\theta_{\tau}(x)$ is a p-adic analogue of the Heaviside function.

In the ordinary case there is an important relation between amplitudes of the open and the closed strings. This relation give a connection on the tree level as follows

$$A_{tree}^{closed}(s, t, u) = \sin\left(\frac{\pi u}{8}\right) A_{tree}^{open}\left(\frac{s}{4}, \frac{t}{4}\right) A_{tree}^{open}\left(\frac{t}{4}, \frac{u}{4}\right), \quad (6)$$

where s, t, u are the Mandelstam variables.

Let F in eq. (5) be a non-discrete totally disconnected and locally compact field and define also the generalized Heaviside function in the form

$$\theta_\tau(\omega) = \frac{1 + \text{sign}_\tau \omega}{2} \quad (7)$$

which is an analogue of the ordinary one.

Now we will consider the amplitude (5) with the characteristic function in one of the following forms:

$$\theta_{\tau[0,1]}(x) = \theta_\tau(x)\theta_\tau(1-x), \quad (8.1)$$

$$\theta_{\tau[0,1]}(x) = \frac{1}{2}(1 + \text{Sign}_\tau x \cdot \text{Sign}_\tau(1-x)), \quad (8.2)$$

$$\theta_{\tau[0,1]}(x) = \frac{1}{2}(\text{Sign}_\tau x + \text{Sign}_\tau(1-x)), \quad (8.3)$$

$$\theta_{\tau[0,1]}(x) = \frac{1}{2}(\text{Sign}_\tau x - \text{Sign}_\tau(-1) \cdot \text{Sign}_\tau(1-x)), \quad \tau = \varepsilon, \quad (8.4)$$

$$\theta_{\tau[0,1]}(x) = \frac{1}{2}(1 - \text{Sign}_\tau(-1) \cdot \text{Sign}_\tau x \cdot \text{Sign}_\tau(1-x)). \quad (8.5)$$

The corresponding amplitudes (5) can be calculated with the help of the general formula

$$B(\pi_1, \pi_2) = \frac{\Gamma(\pi_1)\Gamma(\pi_2)}{\Gamma(\pi_1\pi_2)}, \quad (9)$$

which connects the beta function

$$B(\pi_1, \pi_2) = \int_F \pi_1(x)|x|^{-1} \pi_2(1-x)|1-x|^{-1} dx, \quad (10)$$

where $\pi(x)$ is a multiplicative character with the gamma function defined by an additive character χ

$$\Gamma(\pi) = \int_F \chi(x)\pi(x)|x|^{-1} dx. \quad (11)$$

Consider now the string amplitudes, constructed over the p-adic fields Q_p and their quadratic extension $Q_p(\sqrt{\tau})$, from the point of view of the product formulae (6) which relates amplitudes of closed and open strings in a very simple form. With regard the case $\tau = \varepsilon$, the closed string amplitude defined on the quadratically extended field $K = Q_p(\sqrt{\varepsilon})$, has the form

$$A_{Q_p(\sqrt{\varepsilon})}^{closed}(a, b, c) = \int_{Q_p(\sqrt{\varepsilon})} |x|^{a-1} |1-x|^{b-1} dx = \frac{1-q^{a-1}}{1-q^{-a}} \cdot \frac{1-q^{b-1}}{1-q^{-b}} \cdot \frac{1-q^{c-1}}{1-q^{-c}}, \quad (12)$$

where $q = p^2$. There are no such formulae as simple as (7) for the above constructed open string amplitudes. However, there exists a formula in the following form

$$A_{Q_p(\sqrt{\varepsilon})}^{closed}(a, b, c) = A_{Q_p}^{open, total}(a, b, c) \tilde{A}_{Q_p}(a, b, c), \quad (13)$$

where

$$A_{Q_p}^{open, total}(a, b, c) = \int_{Q_p} |x|^{a-1} |1-x|^{b-1} dx = \frac{1-p^{a-1}}{1-p^{-a}} \cdot \frac{1-p^{b-1}}{1-p^{-b}} \cdot \frac{1-p^{c-1}}{1-p^{-c}} \quad (14)$$

is a p-adic analogue of the totally crossing symmetric Veneziano amplitude.

Furthermore, the p-adic generalization of the N-point tree amplitude for vector particles [in the bosonic case](#), can be proposed in the following form

$$A(\zeta_1 k_1, \dots, \zeta_n k_n) = g^{n-2} \int_{(Q_p)^{n-3}} \theta_{[0, y_{n-1}, \dots, y_{3,1}]}(y) F(\zeta, k, y) \cdot \prod_{3 \leq i < j \leq n-1} |y_i - y_j|_p^{k_i k_j} \prod_{3 \leq i \leq n-1} (|y_i|_p^{k_n k_i} |1 - y_i|_p^{k_2 k_i} dy_i), \quad (15)$$

where $\theta_{[0, y_{n-1}, \dots, y_{3,1}]}(y)$ is a p-adic generalization of the characteristic function of the simplex $0 \leq y_{n-1} \leq \dots \leq y_4 \leq y_3 \leq 1$ and $F(\zeta, k, y)$ is the part of $\exp \sum_{i=j} \left\{ \frac{1}{2} (\zeta_i \zeta_j) / (y_i - y_j)^2 - k_i k_j / y_i - y_j \right\}$ that is multilinear in all the polarization vectors ζ_i .

7. On adelic strings.[3]

The set of all adeles A may be given in the form

$$A = \bigcup_S A(S), \quad A(S) = R \times \prod_{p \in S} Q_p \times \prod_{p \notin S} Z_p. \quad (16)$$

A has the structure of a topological ring.

We recall that quantum amplitudes defined by means of path integral may be symbolically presented as

$$A(K) = \int A(X) \chi\left(-\frac{1}{h} S[X]\right) DX, \quad (17)$$

where K and X denote classical momenta and configuration space, respectively. $\chi(a)$ is an additive character, $S[X]$ is a classical action and h is the Planck constant.

Now we consider simple p-adic and adelic bosonic string amplitudes based on the functional integral (17). The scattering of two real bosonic strings in 26-dimensional space-time at the tree level can be described in terms of the path integral in 2-dimensional quantum field theory formalism as follows:

$$A_\infty(k_1, \dots, k_4) = g_\infty^2 \int DX \exp\left(\frac{2\pi i}{h} S_0[X]\right) \times \prod_{j=1}^4 \int d^2 \sigma_j \exp\left[\frac{2\pi i}{h} k_\mu^{(j)} X^\mu(\sigma_j, \tau_j)\right], \quad (18)$$

where $DX = DX^0(\sigma, \tau) DX^1(\sigma, \tau) \dots DX^{25}(\sigma, \tau)$, $d^2 \sigma_j = d\sigma_j d\tau_j$ and

$$S_0[X] = -\frac{T}{2} \int d^2 \sigma \partial_\alpha X^\mu \partial^\alpha X_\mu \quad (19)$$

with $\alpha = 0, 1$ and $\mu = 0, 1, \dots, 25$. Using the usual procedure one can obtain the crossing symmetric Veneziano amplitude

$$A_\infty(k_1, \dots, k_4) = g_\infty^2 \int_R |x|_\infty^{k_1 k_2} |1-x|_\infty^{k_2 k_3} dx \quad (20)$$

and similarly the Virasoro-Shapiro one for closed bosonic strings.

As p-adic Veneziano amplitude, it was postulated p-adic analogue of (20), i.e.

$$A_p(k_1, \dots, k_4) = g_p^2 \int_{Q_p} |x|_p^{k_1 k_2} |1-x|_p^{k_2 k_3} dx, \quad (21)$$

where only the string world sheet (parametrized by x) is p-adic. Expressions (20) and (21) are Gel'fand-Graev beta functions on R and Q_p , respectively.

Now we take p-adic analogue of (18), i.e.

$$A_p(k_1, \dots, k_4) = g_p^2 \int DX \chi_p \left(-\frac{1}{h} S_0[X] \right) \times \prod_{j=1}^4 \int d^2 \sigma_j \chi_p \left(-\frac{1}{h} k_\mu^{(j)} X^\mu(\sigma_j, \tau_j) \right), \quad (22)$$

to be p-adic string amplitude, where $\chi_p(u) = \exp(2\pi i \{u\}_p)$ is p-adic additive character and $\{u\}_p$ is the fractional part of $u \in Q_p$. In (22), all space-time coordinates X_μ , momenta k_i and world sheet (σ, τ) are p-adic.

Evaluation of (22), in analogous way to the real case, leads to

$$A_p(k_1, \dots, k_4) = g_p^2 \prod_{j=1}^4 \int d^2 \sigma_j \times \chi_p \left[\frac{\sqrt{-1}}{2hT} \sum_{i < j} k_i k_j \log \left((\sigma_i - \sigma_j)^2 + (\tau_i - \tau_j)^2 \right) \right]. \quad (23)$$

Adelic string amplitude is product of real and all p-adic amplitudes, i.e.

$$A_A(k_1, \dots, k_4) = A_\infty(k_1, \dots, k_4) \prod_p A_p(k_1, \dots, k_4). \quad (24)$$

In the case of the Veneziano amplitude and $(\sigma_i, \tau_j) \in A(S) \times A(S)$, where $A(S)$ is defined in (16), we have

$$A_A(k_1, \dots, k_4) = g_\infty^2 \int_R |x|_\infty^{k_1 k_2} |1-x|_\infty^{k_2 k_3} dx \times \prod_{p \in S} g_p^2 \prod_{j=1}^4 \int d^2 \sigma_j \times \prod_{p \notin S} g_p^2. \quad (25)$$

There is the sense to take adelic coupling constant as

$$g_A^2 = |g_\infty|^2 \prod_p |g_p|^2 = 1, \quad 0 \neq g \in Q. \quad (26)$$

Hence, it follows that p-adic effects in the adelic Veneziano amplitude induce discreteness of string momenta and contribute to an effective coupling constant in the form

$$g_{ef}^2 = g_A^2 \prod_{p \in S} \prod_{j=1}^4 \int d^2 \sigma_j \geq 1. \quad (26b)$$

8. Open and closed scalar zeta strings.[4]

The exact tree-level Lagrangian for effective scalar field φ which describes open p-adic string tachyon is

$$L_p = \frac{1}{g^2} \frac{p^2}{p-1} \left[-\frac{1}{2} \varphi \square \varphi + \frac{1}{p+1} \varphi^{p+1} \right], \quad (27)$$

where p is any prime number, $\square = -\partial_t^2 + \nabla^2$ is the D-dimensional d'Alembertian and we adopt metric with signature $(- + \dots +)$.

Now we want to show a model which incorporates the p-adic string Lagrangians in a restricted adelic way. The eq. (27) take the form:

$$L = \sum_{n \geq 1} C_n L_n = \sum_{n \geq 1} \frac{n-1}{n^2} L_n = \frac{1}{g^2} \left[-\frac{1}{2} \phi \sum_{n \geq 1} n^{\frac{\square}{2}} \phi + \sum_{n \geq 1} \frac{1}{n+1} \phi^{n+1} \right]. \quad (27b)$$

Recall that the Riemann zeta function is defined as

$$\zeta(s) = \sum_{n \geq 1} \frac{1}{n^s} = \prod_p \frac{1}{1-p^{-s}}, \quad s = \sigma + i\tau, \quad \sigma > 1. \quad (28)$$

Employing usual expansion for the logarithmic function and definition (28) we can rewrite (27b) in the form

$$L = -\frac{1}{g^2} [1/2 \phi \zeta(\square/2) \phi + \phi + \ln(1-\phi)], \quad (29)$$

where $|\phi| < 1$. $\zeta(\square/2)$ acts as pseudo-differential operator in the following way:

$$\zeta(\square/2) \phi(x) = \frac{1}{(2\pi)^D} \int e^{ixk} \zeta\left(-\frac{k^2}{2}\right) \tilde{\phi}(k) dk, \quad -k^2 = k_0^2 - \vec{k}^2 > 2 + \varepsilon, \quad (30)$$

where $\tilde{\phi}(k) = \int e^{-ikx} \phi(x) dx$ is the Fourier transform of $\phi(x)$.

Dynamics of this field ϕ is encoded in the (pseudo)differential form of the Riemann zeta function. When the d'Alembertian is an argument of the Riemann zeta function we shall call such string a *zeta string*. Consequently, the above ϕ is an open scalar zeta string. The equation of motion for the zeta string ϕ is

$$\zeta(\square/2) \phi = \frac{1}{(2\pi)^D} \int_{k_0^2 - \vec{k}^2 > 2 + \varepsilon} e^{ixk} \zeta\left(-\frac{k^2}{2}\right) \tilde{\phi}(k) dk = \frac{\phi}{1-\phi} \quad (31)$$

which has an evident solution $\phi = 0$.

For the case of time dependent spatially homogeneous solutions, we have the following equation of motion

$$\zeta\left(\frac{-\partial_t^2}{2}\right)\phi(t) = \frac{1}{(2\pi)} \int_{|k_0| > \sqrt{2} + \varepsilon} e^{-ik_0 t} \zeta\left(\frac{k_0^2}{2}\right) \tilde{\phi}(k_0) dk_0 = \frac{\phi(t)}{1 - \phi(t)}. \quad (32)$$

Finally, with regard the open and closed scalar zeta strings, the equations of motion are

$$\zeta(\square/2)\phi = \frac{1}{(2\pi)^D} \int e^{ik} \zeta\left(-\frac{k^2}{2}\right) \tilde{\phi}(k) dk = \sum_{n \geq 1} \theta^{\frac{n(n-1)}{2}} \phi^n, \quad (33)$$

$$\zeta(\square/4)\theta = \frac{1}{(2\pi)^D} \int e^{ik} \zeta\left(-\frac{k^2}{4}\right) \tilde{\theta}(k) dk = \sum_{n \geq 1} \left[\theta^{n^2} + \frac{n(n-1)}{2(n+1)} \theta^{\frac{n(n-1)}{2}-1} (\phi^{n+1} - 1) \right], \quad (34)$$

and one can easily see trivial solution $\phi = \theta = 0$.

9. Mathematical Connections.

Now we take the eq. (1) of **Section 5**. We note that can be related with the Godston-Montgomery equation, the ten dimensional action of IIB supergravity, the relationship of Palumbo-Nardelli model **[1]** and the p-adic, adelic and zeta strings (**Sections 6, 7 and 8**), hence we have the following connections:

$$\begin{aligned}
\delta_k(u) &:= \delta_{k,B}(u) = \left(\frac{1}{\lambda'_{\hat{E}_B}(T)} \frac{d}{dT} \right)^k \log f_{u,B}(T) \Big|_{T=0} \Rightarrow 2 \ln \rho \Rightarrow \int_0^T f(t) dt = (1 + \varepsilon') T \log T, \Rightarrow \\
&\Rightarrow \int d^{10} x \sqrt{|g|} \left[\frac{1}{4} R - \frac{1}{2} (\partial \phi)^2 - \frac{1}{12} e^{-2\phi} H_{\mu\nu\lambda} H^{\mu\nu\lambda} \right] \Rightarrow \\
&\Rightarrow \int_0^\infty \frac{1}{2\kappa_{10}^2} \int d^{10} x (-G)^{1/2} e^{-2\Phi} \left[R + 4 \partial_\mu \Phi \partial^\mu \Phi - \frac{1}{2} |\tilde{H}_3|^2 - \frac{\kappa_{10}^2}{g_{10}^2} Tr_v(|F_2|^2) \right] = \\
&\quad - \int d^{26} x \sqrt{g} \left[-\frac{R}{16\pi G} - \frac{1}{8} g^{\mu\rho} g^{\nu\sigma} Tr(G_{\mu\nu} G_{\rho\sigma}) \right] f(\phi) - \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi \Rightarrow \\
&\Rightarrow A(\zeta_1 k_1, \dots, \zeta_n k_n) = g^{n-2} \int_{(\mathcal{Q}_p)^{n-3}} \theta_{[0, y_{n-1}, \dots, y_{3,1}]}(y) F(\zeta, k, y) \cdot \prod_{3 \leq i < j \leq n-1} |y_i - y_j|_p^{k_i k_j} \prod_{3 \leq i \leq n-1} (|y_i|_p^{k_i k_i} |1 - y_i|^{k_2 k_i} dy_i) \Rightarrow \\
&\Rightarrow A_p(k_1, \dots, k_4) = g_p^2 \int DX \chi_p \left(-\frac{1}{h} S_0[X] \right) \times \prod_{j=1}^4 \int d^2 \sigma_j \chi_p \left(-\frac{1}{h} k_\mu^{(j)} X^\mu(\sigma_j, \tau_j) \right) \Rightarrow \\
&\Rightarrow \zeta(\square/2) \phi = \frac{1}{(2\pi)^D} \int_{k_0^2 - \bar{k}^2 > 2 + \varepsilon} e^{ixk} \zeta \left(-\frac{k^2}{2} \right) \tilde{\phi}(k) dk = \frac{\phi}{1 - \phi}. \quad (1)
\end{aligned}$$

Now we take the eq. (10) of **Section 5**. We note that can be related with the equation regarding the Palumbo-Nardelli model, with p-adic, adelic and zeta strings (**Sections 6,7** and **8**) and with the Ramanujan's identity concerning π [1]. Hence, we have the following connections:

$$\begin{aligned}
\langle f_\varphi, f_\varphi \rangle &= \frac{1}{16\pi^3} N^2 \left\{ \prod_{q|N} \left(1 - \frac{1}{q} \right) \right\} L_N(2, \varphi^2 \bar{\chi}) L_N(1, \psi) \Rightarrow \\
&\Rightarrow \int_0^\infty \pi^2 \langle f_\varphi, f_\varphi \rangle \cdot \frac{1}{N^2 \left\{ \prod_{q|N} \left(1 - \frac{1}{q} \right) \right\} L_N(2, \varphi^2 \bar{\chi}) L_N(1, \psi) G_N} \int d^{10}x (-G)^{1/2} e^{-2\Phi} \cdot \\
&\quad \cdot \left[R + 4\partial_\mu \Phi \partial^\mu \Phi - \frac{1}{2} |\tilde{H}_3|^2 - \frac{\kappa_{10}^2}{g_{10}^2} Tr_\nu (|F_2|^2) \right] = \\
&= -\int d^{26}x \sqrt{g} \left[\left(-R \cdot \pi^2 \langle f_\varphi, f_\varphi \rangle \cdot \frac{1}{N^2 \left\{ \prod_{q|N} \left(1 - \frac{1}{q} \right) \right\} L_N(2, \varphi^2 \bar{\chi}) L_N(1, \psi) G} \right) - \frac{1}{8} g^{\mu\rho} g^{\nu\sigma} Tr(G_{\mu\nu} G_{\rho\sigma}) f(\phi) + \right. \\
&\quad \left. - \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi \right] \Rightarrow \\
&\Rightarrow A(\zeta_1 k_1, \dots, \zeta_n k_n) = g^{n-2} \int_{(\mathcal{Q}_p)^{n-3}} \theta_{[0, y_{n-1}, \dots, y_{3,1}]}(y) F(\zeta, k, y) \cdot \prod_{3 \leq i < j \leq n-1} |y_i - y_j|_p^{k_i k_j} \prod_{3 \leq i \leq n-1} (|y_i|_p^{k_i k_i} |1 - y_i|^{k_2 k_i} dy_i) \Rightarrow \\
&\Rightarrow A_p(k_1, \dots, k_4) = g_p^2 \int DX \mathcal{X}_p \left(-\frac{1}{h} S_0[X] \right) \times \prod_{j=1}^4 \int d^2 \sigma_j \mathcal{X}_p \left(-\frac{1}{h} k_\mu^{(j)} X^\mu(\sigma_j, \tau_j) \right) \Rightarrow \\
&\Rightarrow \zeta(\square/2) \phi = \frac{1}{(2\pi)^D} \int_{k_0^2 - \bar{k}^2 > 2 + \varepsilon} e^{ixk} \zeta \left(-\frac{k^2}{2} \right) \tilde{\phi}(k) dk = \frac{\phi}{1 - \phi}. \quad (2)
\end{aligned}$$

$$\begin{aligned}
\langle f_\varphi, f_\varphi \rangle &= \frac{1}{16\pi^3} N^2 \left\{ \prod_{\substack{q|N \\ q \neq S_\varphi}} \left(1 - \frac{1}{q} \right) \right\} L_N(2, \varphi^2 \bar{\chi}) L_N(1, \psi) \Rightarrow \\
\Rightarrow & \frac{1}{16 \left\{ 2\Phi - \frac{3}{20} \left[R(q) + \frac{\sqrt{5}}{1 + \frac{3 + \sqrt{5}}{2} \exp\left(\frac{1}{\sqrt{5}} \int_0^q \frac{f^5(-t)}{f(-t^{1/5})} t^{4/5} dt \right)} \right] \right\}^3} N^2 \left\{ \prod_{\substack{q|N \\ q \neq S_\varphi}} \left(1 - \frac{1}{q} \right) \right\} L_N(2, \varphi^2 \bar{\chi}) L_N(1, \psi).
\end{aligned}
\tag{3}$$

Furthermore, also here is possible the mathematical connections between π and the Aurea ratio $\phi = \sqrt{5} - 1/2$ by the following simple relationship:

$$\arccos \phi = 0,2879\pi \quad (4)$$

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