Proposal Demonstration of Andrica's Conjecture

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Abstract

In this paper we will show an our short proof of Andrica's conjecture:

$$\sqrt{p_{n+1}} - \sqrt{p_n} < 1$$

using some our results on Legendre's conjecture ([2]). In appendix also an algorithm in PARI/GP to search for counter-examples in a numeric range.

Andrica's conjecture

Andrica 's conjecture is so defined from homonymous voice of Wikipedia:

"…Andrica's Conjecture is a conjecture of Numbers' Theory, concerning the gaps between two successive prime numbers, formulated by romeno's mathematician Dorin Andrica in 1986. It affirms that, for every couple of consecutive numbers p_n and p_{n+1}, we have:

$$\sqrt{p_{n+1}} - \sqrt{p_n} < 1$$

If we pose $g_n = p_{n+1} – p_n$, then the conjecture can be written as

$$g_n < 2\sqrt{p_n} + 1 \text{ ''}.$$

For the rest we send back to homonymous voice of Wikipedia.

Now we propose an our simple proof based on some demonstrations concerning the Legendre's Conjecture, and on square roots of numbers included in the numeric gap between a square and the successive one.

From Wikipedia: "Legendre's Conjecture, by Adrien – Marie Legendre, affirms that exists always a prime number between n^2 and (n+1)^2. This conjecture is one of problems of Landau and, till now, it has not been demonstrated".

Some observations about Legendre's conjectures are:

- between n^2 and (n+1)^2 don't exists always a prime number, but at least two.

- ERATOSTENE Group has demonstrated it, see [1] and [2].

Difference between to perfect squares in the range I = [n^2, (n+1)^2]

To examine the connection between Legendre's conjecture and the Andrica 's conjecture, we must introduce some concepts.

Let be I the closed range of integers definite as I = [n^2, (n+1)^2].

Let be D_{qp} the difference between to consecutive perfect squares, in the range I.

Lemma 1.
The difference D_{qp} between two consecutive perfect squares, in a closed range of integers I = [n^2, (n+1)^2] is always an odd number.

Dem.

D_{qp} = (n +1)^2 – n^2 = 2n +1

Since for every n, D_{qp} = 2n + 1, then is always odd.

Example:

n = 2, valid for all the n natural numbers.

D_{qp} = 3^2 - 2^2 =9 - 4 = 5 = 2*2 +1

Lemma 2.
The number n of integer included in a closed range of integers I = [n^2, (n+1)^2] is even number.

Dem.
For Lemma 1, since number of integer in I is:

N = D_{qp} +1 = 2n + 2 = 2(n + 1), then N is even number.

Example:

If n = 2 N = 2(3) = 6 . Indeed the numbers included in gap I are : 4, 5, 6, 7, 8, 9; with 5 and 7 prime numbers.

Square roots of numbers in gap I = [n^2, (n+1)^2].

Lemma 3.

The difference D_{rq} of square roots of two numbers, also not consecutive (prime and composite), in a closed range of integers I = [n^2, (n+1)^2], it excepts the number (n +1)^2, is smaller than 1.

Dem.

At the extremes of the range of integer, D_{rq} is :

D_{rq} = n – n = 0 if we consider at beginning of the interval the difference with itself or D_{rq} = (n+1) –n = 0. Therefore D_{rq} changes between 0 and 1.

Lemma 3 excludes the numbers (n + 1)^2, because in the second case the difference doesn't give a decimal part after the point. For this Lemma 3 is to check between n^2 and (n +1)^2 -1.

Since we think true the Lengendre's conjecture (see [2]), then between n^2 and (n + 1)^2 exists at least a prime number and therefore a integer, so between the values 0 and 1 assumed by Drq exist some values smaller than 1.

Obviously since we make reference at integer numbers in the range I; it is indifferent that they are prime or composite. Therefore it is possible the applicability of ex - Legendre's conjecture.

Example Square roots for n = 2.

√4 = 2,00

√5 = 2,23 with 5 prime number pn

√6 = 2,44

√7 = 2,64 with 7 prime number pn+1

√8 = 2,82

√9 = 3,00

Lemma 4.

In the range of integers I = [n^2, (n+1)^2] exist at least two prime numbers.

Dem.

The Bertrand's postulate, that is true, says that "If n is an integer with n > 1, then there is always a prime number such that n < p < 2n".

If we call a = n^2 then the space that we are considering is [a, a + 1 + 2√a]. Now for n > 3 the term a + 1 + 2√a > 2a; therefore certainly is applicable the Betrand's postulate but we observe also that this space is bigger than of that used in Bertrand's postulate, therefore it increases the probability to find at least a second prime number; in fact for Prime Number Theorem is:

$$\pi((n+1)^2) - \pi(n^2) \approx \frac{2n+1}{\ln((n+1)^2)} > 1$$

Note: the range that we consider is bigger than the critical smaller space where we could risk don't find the second prime number, but the (1) says that in the case of Andrica's conjecture, we think, moreover, that the two consecutive prime numbers exists also a notable distances or notable gaps.

Example:

In the interval I = n =2 we have the two consecutive prime numbers 5 and 7.

$\sqrt{7} - \sqrt{5}$ = 2,64 – 2.23 = 0,41 < 1

From (1) results $\pi((n+1)^2)-\pi(n^2) \approx 2,27$

Lemma 5.

The difference of square roots of two consecutive prime numbers that are in a closed interval of integers I = [n^2, (n+1)^2], except the number (n+1)^2, is smaller of 1.

Dem.
The Lemma 5 is a consequence of Lemma 3, 4.

It is not still a demonstration of Andrica's conjecture; because the consecutive prime numbers could belong to different square intervals.

Prime numbers in different square intervals.

Some prime numbers belong to successive square interval, also being valid the Legendre's conjecture, for Example 113 an 127: the first is included in space between 10^2 and 11^2, the second 127 is included in the interval between 11^2 and 12^2. Really the square space is always possible individualize only one: for example it is between 10^2 and 12^2.

Lemma 6.

The difference of square roots of two numbers included in a closed interval of integers I = [n^2, (n+1)^2] with k > 1, except the number (n+k)^2, that if k > 1 the difference (n+k)^2 – n^2 $\neq$ 0 mod 3.

Dem.

Lemma 6 can be demonstrated with all previous Lemmas, marking also that k tends only to increase certainly the interval of squares, Therefore the Lemma 6 is a generalization of lemma 5. In particular if k = 1 we return at Lemma 5 and it doesn't occur to consider if the difference is a multiple of 3.

For example 131 and 137 are two prime numbers with their difference is multiple of 3, but it doesn't count because the range is the same for both the prime numbers. In fact is [11^2, 12^2] with k = 1.

Instead if we look 113 and 137 the range to consider is different, that is k = 2 in fact is [10^2, 12^2] but 137 – 113 = 24 multiple of 3.

Here the difference between the square roots is greater than 1 when the difference is even and multiple of 3 (it is the same to say that it is multiple of 6). But there is to say that 137 and 113 are not neither consecutive prime numbers. The problem that the difference between square roots of two consecutive prime numbers can be greater than 1 could happen when the two square intervals are not adjacent, therefore for example for k > 2.

Lemma 7.

Two consecutive prime numbers, included in closed interval of integers I = [n^2, (n+1)^2] with k > 2, haven't got a difference D = 6j when j > 1 (or even and multiple of 3).

Dem.
If the difference of two consecutive prime numbers prime number is even and multiple of 3, it could be:

$$D = 3m = 6j \text{ where } m = 2j \text{ (even)}$$

If j = 1 we have the situation k = 2 and the prime numbers as 23 and 29 where D = 6 (j=1) and the difference of square roots is smaller than 1; therefore the Lemma concentrate itself on cases j > 1: Now the prime numbers can be to build with generator form $p_n = 6n \pm 1$, therefore if there exist two consecutive prime numbers in interval I with k > 2 and j > 1 they have never D = 6j for the same generator form.

However, if for absurd the consecutive prime numbers are such that:

$$p_{n+1} - p_n = 6j, \quad j > 1 \quad (2)$$

Equivalent to:

$$\sqrt{p_{n+1}} = \sqrt{p_n + 6j}$$

Then we conclude that

$$\sqrt{p_{n+1}} - \sqrt{p_n} = \sqrt{p_n + 6j} - \sqrt{p_n} > 1 \text{ (3)}$$

Since the (2) is false, we cannot conclude the (3). In other words if the difference between consecutive prime numbers is multiple of 6we have always found that the difference of square roots of two consecutive prime numbers is greater than 1.

Andrica conjecture

The Andrica's conjecture is true for consequence of Lemma 6 and Lemma 7.

Dem.

The conjecture supposes the existence of two consecutive prime numbers. If these are included in the same square interval already the Lemma 5 will gives true the conjecture. If the two prime numbers are included in different square intervals then Lemma 6 and Lemma 7 guarantee that the conjecture is true.

Appendix.

Here we present two programs in PARI/GP that search counter-example about Andrica's conjecture. The second program searches on the fact that in an square interval with k > 2 two prime numbers if consecutive cannot to have a difference even and multiple of 3.

Both the programs have the advantage that can be launched with a initial value and a final one of search, permitting to take up again the search, on successive intervals The authors have tested between 1 and one milliard without to find counter – examples.

```
sT(val1=1, val2=1000000000) = local(); {
  print("Ricerca controesempi congettura Andrica - vers. 1.00 FEB 2010\n");
  print("Numero di partenza:      ",val1);
  print("Numero di arresto:      ",val2);
  p1=nextprime(val1);
  d=0;
  while(d<1 & p1 < val2,
    p2=nextprime(p1+1);
    d=sqrt(p2)-sqrt(p1);
    print("d:                  ",d);
    if(
      d > 1,
      print("controesempio !!!");
      print("Pn:                 ",p1);
      print("Pn+1:               ",p2);
    );
    p1=p2;
  );
}


sT2(val1=1, val2=100000) = local(); {

  print("Ricerca controesempi Differenza primi consecutivi non multipli di 3 - vers. 1.00 FEB 2010\n");
  print("Numero di partenza:      ",val1);
  print("Numero di arresto:      ",val2);
  a=1;
  p1=nextprime(val1);
  while(p1 < val2,
    p2=nextprime(p1+1);
    d=p2-p1;
    print("**********************");
    print("Pn:                 ",p1);
    print("Pn+1:               ",p2);
    print("d:                  ",d);
    a = d%3;
    a1 = sqrtint(p1);
    if ( a1 == p1, a1 = a1+1;);
    a2 = sqrtint(p2);
    while(a2^2<p2,
        a2=a2+1;
    );

    print("I =[",a1^2,",",a2^2,"]");
    k=a2-a1;
    print("k:                              ",k);
    c1=p1;
    p1=p2;
    if(a==0 & k>1,
      print("controesempio !!!");
      print("Pn:                 ",c1);
      print("Pn+1:               ",p2);
      print("a1:                             ",a1);
      print("a2:                             ",a2);
      p1=val2;
```

```
  );
}
```

References

[1] "Unified solution for some conjectures on number of prime numbers in a certain interval
Prof Annarita Tulumello, in section "Articles on prime numbers"

[2]"The Landau'd prime numbers and the Legendre's Conjecture" (La congettura di Landau e la congettura di Legendre) di Rosario Turco, Maria Colonnese, Michele Nardelli, Giovanni Di Maria, Francesco Di Noto, Annarita Tulumello

[3] "Prime numbers in search of Author", Rosario Turco, Maria Colonnese, Giovanni Di Maria, Francesco Di Noto, Michele Nardelli, Annarita Tulumello.

[4] " Proposal of solution of Andrica's Conjecture" Rosario Turco, Maria Colonnese, Giovanni Di Maria, Francesco Di Noto, Michele Nardelli, Annarita Tulumello.