

CUT/P

**Una nota sullo sviluppo della derivata di ordine n (n intero positivo)
delle funzioni trigonometriche $P(x) = \tan(x)$, e $C(x) = \sec(x)$.**

Considerazioni ed osservazioni

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**Development of derivatives of order n (n positive integer) of the trigonometric
functions $P(x) = \tan(x)$ and $C(x) = \sec(x)$. Some remarks by
Pasquale Cutolo**

Introduzione-

Con il presente studio l'autore, utilizzando formule note, esamina lo sviluppo delle derivate, di ordine $(2n-1)$, e di ordine $(2n)$, ($n = 1,2,3,\dots$), delle funzioni trigonometriche $P(x) = \tan(x)$, e $C(x) = \sec(x)$, determinando le espressioni che definiscono i coefficienti, in funzione di n , dei polinomi risultanti, ottenendo formule complesse straordinarie ed inusuali. Alla fine viene aggiunta una interessante applicazione.

Abstract :

In this report the author uses well known formulas to revisit the development of the derivatives of order $(2n-1)$ and $(2n)$, ($n = 1,2,3,\dots$), of the trigonometric functions $P(x) = \tan(x)$ and $C(x) = \sec(x)$.

The formulas for the functions of n giving the coefficients of the resulting polynomial expansions are derived through unusual developments of impressive complexity. In the conclusions an interesting application is presented.

A.0.0- Caso A – P(x) = Tan(x)

E' nota la seguente relazione dei complementi:

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\text{Sin}(\pi z)}, \quad 0 < z < 1 \quad (\text{A.0.0.01})$$

essendo, $\Gamma(z) = \int_0^{\infty} t^{z-1} e^{-z} dt$, $\text{Re}(z) > 0$, la ben nota funzione di Eulero di seconda specie.Ponendo, nella (A.0.0.01), $z = \frac{1}{2} - x$, (x reale), e prendendo il logaritmo naturale

di ambo i membri della (A.0.0.01), otteniamo:

$$\ln \Gamma\left(\frac{1}{2} - x\right) + \ln \Gamma\left(\frac{1}{2} + x\right) = \ln \pi - \ln \text{Cos}(\pi x) \quad (\text{A.0.0.02})$$

Derivando, rispetto ad x, i due membri della (A.0.0.02), ricaviamo:

$$\Psi\left(\frac{1}{2} + x\right) - \Psi\left(\frac{1}{2} - x\right) = \pi \text{Tan}(\pi x), \quad (\text{A.0.0.03})$$

essendo $\Psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}$, e $\text{Tan}(\pi x)$ è la tangente trigonometrica di (πx) .Ricordando che $\Psi(x) = \int_0^1 \frac{t^{x-1} - 1}{t-1} dt + \gamma$, (Ved. [1], 9.541, pag. 1074), essendo $\gamma = 0,5772156649$ la costante di Eulero-Mascheroni, dalla (A.0.0.03) ricaviamo:

$$\begin{aligned} \int_0^1 \frac{t^{x-1/2} - t^{-x-1/2}}{t-1} dt &= (t = e^{-u}) = \int_0^{\infty} \frac{e^{-u(x-1/2)} - e^{-u(-x-1/2)}}{e^{-u} - 1} e^{-u} du = \int_0^{\infty} \frac{(e^{ux} - e^{-ux})e^{u/2}}{e^u - 1} du = \\ &= \int_0^{\infty} \frac{\text{Sinh}(ux)}{\text{Sinh}(u/2)} du = \pi \text{Tan}(\pi x), \quad |x| < \frac{1}{2}, \end{aligned} \quad (\text{A.0.0.04})$$

A.1.0- Derivata di ordine (2n-1) della funzione P(x) = Tan(x).

Derivando, (2n-1) volte, la (A.0.0.04), rispetto ad x, troviamo:

$$\begin{aligned} \int_0^{\infty} \frac{u^{2n-1} \text{Cosh}(ux)}{\text{Sinh}(u/2)} du &= [\pi \text{Tan}(\pi x)]^{(2n-1)} = \pi^{2n} \sum_{h=0}^n a_h [\text{Tan}(\pi x)]^{2h} \\ &= \frac{-2\pi}{i} \left[\sum_{k \geq 0} (-1)^k e^{-2\pi i x(1+k)} \right]^{(2n-1)} \end{aligned} \quad (\text{A.1.0.01})$$

Non è difficile verificare la relazione:

$$[\pi \text{Tan}(\pi x)]^{(2n-1)} = \pi^{2n} \sum_{h=0}^n a_h [\text{Tan}(\pi x)]^{2h} \quad (\text{A.1.0.02})$$

e la relazione $[\text{Tan}(x)]^{(2n-1)} = \sum_{h=0}^n a_h [\text{Tan}(x)]^{2h}$ **A.1.1 - Determinazione del termine noto a_0 del polinomio rappresentato**dallo sviluppo della funzione $[\text{Tan}(x)]^{(2n-1)} = \sum_{h=0}^n a_h [\text{Tan}(x)]^{2h}$

Dalla (A.1.0.01) ricaviamo:

$$\begin{aligned} [\pi \operatorname{Tan}(\pi x)]^{(2n-1)} &= \left[\frac{\pi}{i} \left(1 - \frac{2}{e^{2\pi i x} + 1} \right) \right]^{(2n-1)} = \frac{-2\pi}{i} \left[\sum_{k \geq 0} (-1)^k e^{-2\pi i x(1+k)} \right]^{(2n-1)} = \\ &= \frac{-2\pi}{i} \sum_{k \geq 0} (-1)^k [-2\pi i(1+k)]^{2n-1} e^{-2\pi i x(1+k)}, \end{aligned} \quad (\text{A.1.1.01})$$

$i = \sqrt{-1}$; sostituendo k a $(1+k)$, abbiamo:

$$[\pi \operatorname{Tan}(\pi x)]^{(2n-1)} = 2^{2n} \pi^{2n} (-1)^n \sum_{k \geq 1} (-1)^k k^{2n-1} e^{-2\pi i x k} \quad (\text{A.1.1.02})$$

Ponendo, nella (A.1.0.01) e (A.1.1.02), $x = 0$, troviamo:

$$\int_0^\infty \frac{u^{2n-1}}{\operatorname{Sinh}(u/2)} du = \pi^{2n} a_0 = 2^{2n} \pi^{2n} (-1)^n \sum_{k \geq 1} (-1)^k k^{2n-1} \quad (\text{A.1.1.03})$$

Osserviamo che:

$$\begin{aligned} \sum_{k \geq 1} (-1)^k k^{2n-1} &= \sum_{k \geq 1} (2k)^{2n-1} - \sum_{k \geq 1} (2k-1)^{2n-1} = \\ &= \sum_{k \geq 1} (2k)^{2n-1} - \left[\sum_{k \geq 1} (2k-1)^{2n-1} + \sum_{k \geq 1} (2k)^{2n-1} - \sum_{k \geq 1} (2k)^{2n-1} \right] = \\ &= \sum_{k \geq 1} (2k)^{2n-1} - \sum_{k \geq 1} k^{2n-1} + \sum_{k \geq 1} (2k)^{2n-1} = (2^{2n} - 1) \zeta(1-2n); \end{aligned} \quad (\text{A.1.1.04})$$

ricordando che $\zeta(1-2n) = -\frac{B_{2n}}{2n}$, e $\zeta(-2n) = 0$, (Ved. [1], 9.541, pag. 1074),

$$\text{troviamo che: } \sum_{k \geq 1} (-1)^k k^{2n-1} = (2^{2n} - 1) \left(-\frac{B_{2n}}{2n} \right) \quad (\text{A.1.1.05})$$

essendo B_{2n} il numero di Bernoulli di indice $2n$, mentre $\zeta(s)$ è la funzione

$$\text{Zeta di Riemann, definita da: } \zeta(s) = \sum_{k \geq 1} \frac{1}{k^s}, \operatorname{Re}(s) > 1$$

Sostituendo il risultato della (A.1.1.05) nella (A.1.1.03), otteniamo:

$$\pi^{2n} a_0 = 2^{2n} \pi^{2n} (-1)^{n-1} (2^{2n} - 1) \frac{B_{2n}}{2n}$$

Ricordando, inoltre, che $(-1)^{n-1} B_{2n} = |B_{2n}|$, e semplificando, abbiamo:

$$a_0 = 2^{2n} (-1)^{n-1} (2^{2n} - 1) \frac{B_{2n}}{2n} = 2^{2n} (2^{2n} - 1) \frac{|B_{2n}|}{2n} \quad (\text{A.1.1.06})$$

che rappresenta l'espressione del termine noto, in funzione di n , del polinomio

$$\sum_{h=0}^n a_h [\operatorname{Tan}(x)]^{2h}, \text{ di grado } 2n.$$

Osserviamo che, per n intero positivo, a_0 risulta sempre intero positivo;

poiché $|B_{2n}|$ è dato dal rapporto di due numeri interi, indicando con p e q

detti numeri interi, ($|B_{2n}| = p/q$), l'espressione (A.1.1.06) indica che a_0 rappresenta un numero che è un multiplo intero del numero intero q e di $2n$.

L'espressione di a_0 possiamo ricavarla anche utilizzando la relazione (A.1.1.03), infatti:

$$\int_0^\infty \frac{u^{2n-1}}{\operatorname{Sinh}(u/2)} du = \pi^{2n} a_0 = 2 \int_0^\infty u^{2n-1} e^{-u/2} \sum_{k \geq 0} e^{-uk} du = 2 \sum_{k \geq 0} \frac{(2n-1)!}{\left(\frac{1}{2} + k\right)^{2n}} =$$

$$= 2^{2n+1} (2n-1)! \sum_{k \geq 0} \frac{1}{(1+2k)^{2n}}; \text{ ma,}$$

$$\sum_{k \geq 0} \frac{1}{(1+2k)^{2n}} = \sum_{k \geq 0} \frac{1}{(1+2k)^{2n}} + \sum_{k \geq 1} \frac{1}{(2k)^{2n}} - \sum_{k \geq 1} \frac{1}{(2k)^{2n}} =$$

$$= \sum_{k \geq 1} \frac{1}{(k)^{2n}} - \sum_{k \geq 1} \frac{1}{(2k)^{2n}} = (1-2^{-2n}) \zeta(2n); \quad (\text{A.1.1.07})$$

ricordando che: $\zeta(2n) = \frac{2^{2n-1} \pi^{2n}}{(2n)!} |B_{2n}|,$ (A.1.1.08)

(Ved. [1], 9.541, pag. 1074), otteniamo:

$$2^{2n+1} (2n-1)! \sum_{k \geq 0} \frac{1}{(1+2k)^{2n}} = 2^{2n+1} (2n-1)! (1-2^{-2n}) \frac{2^{2n-1} \pi^{2n}}{(2n)!} |B_{2n}| = \pi^{2n} a_0, \quad (\text{A.1.1.09})$$

da cui:
$$a_0 = 2^{2n} (2^{2n} - 1) \frac{|B_{2n}|}{2n}$$

La relazione (A.1.1.06) è stata verificata con un programma di matematica.

Riportiamo i valori di a_0 per n variabile da 1 a 10.

n = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10
$a_0 = 1, 2, 16, 272, 7936, 353792, 22368256, 1903757312, 209865342976, 29088885112832$

Dalla (A.1.1.09) ricaviamo:

$$\sum_{k \geq 0} \frac{1}{(1+2k)^{2n}} = \frac{(2^{2n} - 1) \pi^{2n}}{2(2n)!} |B_{2n}| \quad (\text{A.1.1.10})$$

A.1.2. – Determinazione dell’espressione della Somma $\sum_{h=0}^n a_h$, in funzione di n.

Utilizzando le relazioni (A.1.0.01) e (A.1.1.02), e ponendo, $x = 1/4$, troviamo:

$$\pi^{2n} \sum_{h=0}^n a_h = 2^{2n} \pi^{2n} (-1)^n \sum_{k \geq 1} (-1)^k k^{2n-1} e^{-2\pi i k / 4} = 2^{2n} \pi^{2n} (-1)^n \sum_{k \geq 1} (-1)^k k^{2n-1} (-i)^k, \text{ cioè:}$$

$$\sum_{h=0}^n a_h = 2^{2n} (-1)^n \sum_{k \geq 1} (i)^k k^{2n-1} = 2^{2n} (-1)^n \left[\sum_{k \geq 1} (-1)^k (2k)^{2n-1} + \sum_{k \geq 1} (i)^{2k-1} (2k-1)^{2n-1} \right];$$

uguagliando le parti reali della precedente relazione, troviamo:

$$\sum_{h=0}^n a_h = 2^{2n} (-1)^n \sum_{k \geq 1} (-1)^k (2k)^{2n-1};$$

utilizzando la (A.1.1.04), o la (A.1.1.05), abbiamo:

$$\sum_{h=0}^n a_h = 2^{2n} 2^{2n-1} (-1)^n (2^{2n} - 1) \zeta(1-2n) = 2^{4n-1} (-1)^{n-1} (2^{2n} - 1) \frac{B_{2n}}{2n} =$$

$$= 2^{4n-1} (2^{2n} - 1) \frac{|B_{2n}|}{2n}, \quad (\text{A.1.2.01})$$

La relazione (A.1.2.01) è stata verificata con un programma di matematica.

Riportiamo i primi valori di $\sum_{h=0}^n a_h$ per n variabile da 1 a 10.

n = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10

$$\sum_{h=0}^n a_h = 2, 16, 512, 34816, 4063232, 724566016, 183240753152, 62382319599616, \\ 27507470234550272, 15250953398036463616$$

Dividendo il risultato della (A.1.2.01) con quello della (A.1.1.06), otteniamo:

$$\left(\sum_{h=0}^n a_h\right)/a_0 = 2^{2n-1} \quad (\text{A.1.2.02})$$

A.1.3 -Determinazione delle espressioni che definiscono, in funzione di n,

i coefficienti $a_1, a_2, a_3, \dots, a_n$ dei termini del polinomio $\sum_{h=0}^n a_h [\text{Tan}(x)]^{2h}$

Riprendiamo la formula indicata con (A.1.1.02)

$$[\pi \text{Tan}(\pi x)]^{(2n-1)} = \pi^{2n} \sum_{h=0}^n a_h [\text{Tan}(\pi x)]^{2h} = 2^{2n} \pi^{2n} (-1)^n \sum_{k \geq 1} (-1)^k k^{2n-1} e^{-2\pi i x k}, \quad (\text{A.1.3.01})$$

e poniamo $\text{Tan}(\pi x) = t$, da cui: $\pi x = \text{ArcTan}(t) = \frac{1}{2i} \ln \frac{1+it}{1-it}$,

che sostituita nella (A.1.3.01) fornisce:

$$\sum_{h=0}^n a_h t^{2h} = 2^{2n} (-1)^n \sum_{k \geq 1} (-1)^k k^{2n-1} e^{-k \ln \frac{1+it}{1-it}} = 2^{2n} (-1)^n \sum_{k \geq 1} (-1)^k k^{2n-1} \left(\frac{1-it}{1+it}\right)^k; \quad (\text{A.1.3.02})$$

derivando la (A.1.3.02), (2h) volte, rispetto a t, e ponendo dopo, t = 0, abbiamo:

$$a_h (2h)! = [2^{2n} (-1)^n \sum_{k \geq 1} (-1)^k k^{2n-1} \left(\frac{1-it}{1+it}\right)^k]_{t=0}^{(2h)};$$

$$\begin{aligned} \text{ora, } [(1-it)^k (1+it)^{-k}]_{t=0}^{(2h)} &= \lim_{t \rightarrow 0} \sum_{j=0}^{2h} \binom{2h}{j} [(1-it)^k]^{(2h-j)} [(1+it)^{-k}]^{(j)} = \\ &= \sum_{j=0}^{2h} \binom{2h}{j} \frac{\Gamma(k+1)(-i)^{2h-j}}{\Gamma(k+1-2h+j)} \frac{\Gamma(k+j)(-i)^j}{\Gamma(k)} = \sum_{j=0}^{2h} \binom{2h}{j} k (-1)^h \frac{\Gamma(k+j)}{\Gamma(k+1-2h+j)} = \\ &= \sum_{j=0}^{2h} \binom{2h}{j} k (-1)^h (k+j-1)(k+j-2)\dots(k+j-2h+1); \end{aligned}$$

ricordiamo che nello sviluppo delle derivate abbiamo utilizzate le formule seguenti:

$$D_x^{(n)} (a+bx)^\alpha = \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+1-n)} b^n (a+bx)^{\alpha-n}; \quad D_x^{(n)} (a+bx)^{-\beta} = \frac{\Gamma(n+\beta)}{\Gamma(\beta)} (-b)^n (a+bx)^{-\beta-n},$$

(Ved. [5]);

inoltre, ricordiamo la ben nota relazione:

$$x(x-1)(x-2)\dots(x-n+1) = \sum_{k=0}^n s(n,k) x^k, \quad (\text{A.1.3.03})$$

dove i coefficienti s(n,k) rappresentano i numeri di Stirling di prima specie.

(Ved. [8])

$$\begin{aligned} \text{Pertanto: } \sum_{j=0}^{2h} \binom{2h}{j} k (-1)^h (k+j-1)(k+j-2)\dots(k+j-2h+1) &= \\ = \sum_{j=0}^{2h} \binom{2h}{j} k (-1)^h \sum_{u=0}^{2h-1} s(2h-1,u) (k+j-1)^u &= \sum_{j=0}^{2h} \binom{2h}{j} k (-1)^h \sum_{u=0}^{2h-1} s(2h-1,u) \sum_{p \geq 0} \binom{u}{p} k^p (j-1)^{u-p}; \end{aligned}$$

quindi: $a_h(2h)! = 2^{2n} (-1)^n \sum_{k \geq 1} (-1)^k k^{2n} \sum_{j=0}^{2h} \binom{2h}{j} (-1)^h \sum_{u=0}^{2h-1} s(2h-1, u) \sum_{p \geq 0} \binom{u}{p} k^p (j-1)^{u-p} =$
 $= 2^{2n} (-1)^n (-1)^h \sum_{j=0}^{2h} \binom{2h}{j} \sum_{u=0}^{2h-1} s(2h-1, u) \sum_{p \geq 0} \binom{u}{p} (j-1)^{u-p} \sum_{k \geq 1} (-1)^k k^{2n+p};$

sappiamo che $\zeta(-2n-2p) = 0$, [(n + p) intero > 0)], e quindi:

$$a_h(2h)! = 2^{2n} (-1)^n (-1)^h \sum_{j=0}^{2h} \binom{2h}{j} \sum_{u=0}^{2h-1} s(2h-1, u) \sum_{p \geq 1} \binom{u}{2p-1} (j-1)^{u-2p+1} \sum_{k \geq 1} (-1)^k k^{2n+2p-1};$$

ricordiamo che:

$$\sum_{k \geq 1} (-1)^k k^{2n+2p-1} = (2^{2n+2p} - 1) \zeta(1-2n-2p) = (2^{2n+2p} - 1) \left(-\frac{B_{2n+2p}}{2n+2p} \right);$$

quindi: $a_h(2h)! =$

$$= 2^{2n} (-1)^{n-1} (-1)^h \sum_{j=0}^{2h} \binom{2h}{j} \sum_{u=0}^{2h-1} s(2h-1, u) \sum_{p \geq 1} \binom{u}{2p-1} (j-1)^{u-2p+1} (2^{2n+2p} - 1) \frac{B_{2n+2p}}{2n+2p}; \quad (\text{A.1.3.04})$$

osserviamo che nello sviluppo della relazione precedente (A.1.3.04), al variare di j, u, p, si presentano valori indeterminati (0^0). Per eliminare detti inconvenienti calcoliamo

l'espressione (A.1.3.04), per j = 0, per j = 1, e per j > 1.

Così operando, troviamo:

$$1) \text{ per } j = 0, \sum_{j=0}^{2h} \binom{2h}{j} k (-1)^h (k+j-1)(k+j-2) \dots (k+j-2h+1) =$$

$$= (-1)^h k(k-1)(k-2) \dots (k-2h+1) = (-1)^h \sum_{u=0}^{2h} s(2h, u) k^u;$$

$$[a_h(2h)!]_{j=0} = 2^{2n} (-1)^n \sum_{k \geq 1} (-1)^k k^{2n-1} (-1)^h \sum_{u=0}^{2h} s(2h, u) k^u =$$

$$= 2^{2n} (-1)^n (-1)^h \sum_{u=0}^{2h} s(2h, u) \sum_{k \geq 1} (-1)^k k^{2n+u-1} =$$

$$= 2^{2n} (-1)^n (-1)^h \sum_{u=0}^h s(2h, 2u) \sum_{k \geq 1} (-1)^k k^{2n+2u-1} =$$

$$= 2^{2n} (-1)^n (-1)^h \sum_{u=0}^h s(2h, 2u) (2^{2n+2u} - 1) \zeta(1-2n-2u) =$$

$$= 2^{2n} (-1)^{n-1} (-1)^h \sum_{u=0}^h s(2h, 2u) (2^{2n+2u} - 1) \frac{B_{2n+2u}}{2n+2u}$$

$$2) \text{ per } j = 1, \sum_{j=0}^{2h} \binom{2h}{j} k (-1)^h (k+j-1)(k+j-2) \dots (k+j-2h+1) =$$

$$= 2hk(-1)^h (k)(k-1) \dots (k-2h+2) = 2hk(-1)^h \sum_{u=0}^{2h-1} s(2h-1, u) k^u;$$

$$a_h(2h)!_{j=1} = 2^{2n} (-1)^n \sum_{k \geq 1} (-1)^k k^{2n-1} 2hk(-1)^h \sum_{u=0}^{2h-1} s(2h-1, u) k^u =$$

$$= 2^{2n+1} (-1)^n h(-1)^h \sum_{u=0}^{2h-1} s(2h-1, u) \sum_{k \geq 1} (-1)^k k^{2n+u} =$$

$$= 2^{2n+1} (-1)^n h(-1)^h \sum_{u=1}^h s(2h-1, 2u-1) \sum_{k \geq 1} (-1)^k k^{2n+2u-1}; \text{ pertanto:}$$

$$\begin{aligned}
a_h(2h)! &= 2^{2n}(-1)^{n-1}(-1)^h \sum_{u=0}^h s(2h,2u)(2^{2n+2u}-1) \frac{B_{2n+2u}}{2n+2u} + \\
&\quad + 2^{2n+1}(-1)^n h(-1)^h \sum_{u=1}^h s(2h-1,2u-1) \sum_{k \geq 1} (-1)^k k^{2n+2u-1} + \\
&\quad + 2^{2n}(-1)^{n-1}(-1)^h \sum_{j=2}^{2h} \binom{2h}{j} \sum_{u=0}^{2h-1} s(2h-1,u) \sum_{p \geq 1} \binom{u}{2p-1} (j-1)^{u-2p+1} (2^{2n+2p}-1) \frac{B_{2n+2p}}{2n+2p}.
\end{aligned}$$

In definitiva, ricaviamo:

$$\begin{aligned}
a_h &= \frac{2^{2n}}{(2h)!} (-1)^{n-1} (-1)^h \left[\sum_{u=0}^h s(2h,2u)(2^{2n+2u}-1) \frac{B_{2n+2u}}{2n+2u} + \right. \\
&\quad \left. + 2h \sum_{u=1}^h s(2h-1,2u-1)(2^{2n+2u}-1) \frac{B_{2n+2u}}{2n+2u} + \right. \\
&\quad \left. + \sum_{j=2}^{2h} \binom{2h}{j} \sum_{u=0}^{2h-1} s(2h-1,u) \sum_{p=1}^{\lfloor (u+1)/2 \rfloor} \binom{u}{2p-1} (j-1)^{u-2p+1} (2^{2n+2p}-1) \frac{B_{2n+2p}}{2n+2p} \right]. \quad (\text{A.1.3.05})
\end{aligned}$$

Ponendo, $h = 0$, nella (A.1.3.05), ritroviamo la formula (A.1.1.06):

$$\text{cioè: } a_0 = 2^{2n}(2^{2n}-1) \frac{|B_{2n}|}{2n}$$

Per $h = n$, ricaviamo:

$$\begin{aligned}
a_n &= -\frac{2^{2n}}{(2n)!} \left[\sum_{u=0}^n s(2n,2u)(2^{2n+2u}-1) \frac{B_{2n+2u}}{2n+2u} + \right. \\
&\quad \left. + 2n \sum_{u=1}^n s(2n-1,2u-1)(2^{2n+2u}-1) \frac{B_{2n+2u}}{2n+2u} + \right. \\
&\quad \left. + \sum_{j=2}^{2n} \binom{2n}{j} \sum_{u=0}^{2n-1} s(2n-1,u) \sum_{p=1}^{\lfloor (u+1)/2 \rfloor} \binom{u}{2p-1} (j-1)^{u-2p+1} (2^{2n+2p}-1) \frac{B_{2n+2p}}{2n+2p} \right] = (2n-1)! \quad (\text{A.1.3.06})
\end{aligned}$$

Le relazioni (A.1.3.05) e (A.1.3.06) sono state verificate con un programma di matematica. E' straordinario verificare che un'espressione molto complessa, come quella di cui al 2° membro della (A.1.3.06), risulta uguale a $(2n-1)!$

A.1.4 – Osserviamo che ponendo, $x = 1/4$, nella (A.1.0.01), troviamo:

$$\int_0^{\infty} \frac{u^{2n-1} \text{Cosh}(u/4)}{\text{Sinh}(u/2)} du = \pi^{2n} \sum_{h=0}^n a_h, \quad (\text{A.1.4.01})$$

Sviluppando i calcoli, abbiamo:

$$\begin{aligned}
&\int_0^{\infty} \frac{u^{2n-1} \text{Cosh}(u/4)}{\text{Sinh}(u/2)} du = \int_0^{\infty} u^{2n-1} (e^{u/4} + e^{-u/4}) \sum_{k \geq 0} e^{-u/2} e^{-uk} du = \\
&= \int_0^{\infty} \sum_{k \geq 0} e^{-uk} u^{2n-1} (e^{-u/4} + e^{-3u/4}) du = (2n-1)! \sum_{k \geq 0} \left[\frac{1}{(\frac{1}{4} + k)^{2n}} + \frac{1}{(\frac{3}{4} + k)^{2n}} \right] = \\
&= (2n-1)! 4^{2n} \sum_{k \geq 0} \left[\frac{1}{(1+4k)^{2n}} + \frac{1}{(3+4k)^{2n}} \right] = \pi^{2n} \sum_{h=0}^n a_h, \quad (\text{A.1.4.02})
\end{aligned}$$

da cui, tenendo presente la (A.1.2.01), ricaviamo:

$$\sum_{k \geq 0} \left[\frac{1}{(1+4k)^{2n}} + \frac{1}{(3+4k)^{2n}} \right] = \frac{\pi^{2n}}{2(2n)!} (2^{2n} - 1) |B_{2n}| \quad (\text{A.1.4.03})$$

Inoltre, dalla (A.1.4.01) otteniamo:

$$\int_0^\infty \frac{u^{2n-1} \text{Cosh}(u/4)}{\text{Sinh}(u/2)} du = \int_0^\infty \frac{u^{2n-1}}{2 \text{Sinh}(u/4)} du = \int_0^\infty u^{2n-1} \sum_{k=0}^\infty e^{-u/4} e^{-uk/2} du =$$

$$= \sum_{k=0}^\infty \frac{(2n-1)!}{\left(\frac{1}{4} + \frac{k}{2}\right)^{2n}} = 4^{2n} (2n-1)! \sum_{k=0}^\infty \frac{1}{(1+2k)^{2n}} = \pi^{2n} \sum_{h=0}^n a_h, \text{ da cui:}$$

$$\sum_{k=0}^\infty \frac{1}{(1+2k)^{2n}} = \frac{\pi^{2n}}{2(2n)!} (2^{2n} - 1) |B_{2n}| \quad (\text{A.1.4.04})$$

che è perfettamente identica alla (A.1.1.10).

Dal confronto tra la (A.1.4.04) e la (A.1.4.03), troviamo, com'era naturale, che:

$$\sum_{k=0}^\infty \frac{1}{(1+2k)^{2n}} = \sum_{k \geq 0} \left[\frac{1}{(1+4k)^{2n}} + \frac{1}{(3+4k)^{2n}} \right] \quad (\text{A.1.4.05})$$

A.1.5 – Considerando il caso generale fornito dalla (A.1.0.01), abbiamo:

$$\int_0^\infty \frac{u^{2n-1} \text{Cosh}(ux)}{\text{Sinh}(u/2)} du = \pi^{2n} \sum_{h=0}^n a_h [\text{Tan}(\pi x)]^{2h} = 2^{2n} \pi^{2n} (-1)^n \sum_{k \geq 1} (-1)^k k^{2n-1} e^{-2\pi i k x}; \quad (\text{A.1.5.01})$$

Dall'integrale del 1° membro della (A.1.5.01), ricaviamo:

$$\int_0^\infty \frac{u^{2n-1} \text{Cosh}(ux)}{\text{Sinh}(u/2)} du = \int_0^\infty u^{2n-1} (e^{ux} + e^{-ux}) \sum_{k \geq 0} e^{-u/2} e^{-uk} du =$$

$$= (2n-1)! \sum_{k \geq 0} \left[\frac{1}{\left(\frac{1}{2} - x + k\right)^{2n}} + \frac{1}{\left(\frac{1}{2} + x + k\right)^{2n}} \right] = \pi^{2n} \sum_{h=0}^n a_h [\text{Tan}(\pi x)]^{2h}, \quad |x| < \frac{1}{2}; \quad (\text{A.1.5.02})$$

$$\text{da cui: } \sum_{k \geq 0} \left[\frac{1}{\left(\frac{1}{2} - x + k\right)^{2n}} + \frac{1}{\left(\frac{1}{2} + x + k\right)^{2n}} \right] = \frac{\pi^{2n}}{(2n-1)!} \sum_{h=0}^n a_h [\text{Tan}(\pi x)]^{2h}; \quad (\text{A.1.5.03})$$

ricordiamo che $a_n = (2n-1)!$;

dalla (1.5.02), per $x = 0$, ritroviamo la formula:

$$2^{2n+1} (2n-1)! \sum_{k \geq 0} \frac{1}{(1+2k)^{2n}} = \pi^{2n} a_0 = \pi^{2n} 2^{2n} (2^{2n} - 1) \frac{|B_{2n}|}{2n}; \quad (\text{A.1.5.04})$$

$$\text{cioè:} \quad \sum_{k \geq 0} \frac{1}{(1+2k)^{2n}} = \frac{(2^{2n} - 1) \pi^{2n}}{2(2n)!} |B_{2n}| \quad (\text{A.1.5.05})$$

identica alla (A.1.1.10).

Per $x = 1/4$, ritroviamo la formula (A.1.4.03)).

Dall'ultimo membro della (A.1.5.01), ricaviamo:

$$\begin{aligned} 2^{2n} \pi^{2n} (-1)^n \sum_{k \geq 1} (-1)^k k^{2n-1} e^{-2\pi i x k} &= 2^{2n} \pi^{2n} (-1)^n \sum_{k \geq 1} (-1)^k k^{2n-1} \sum_{h \geq 0} \frac{(-2i\pi x k)^h}{h!} = \\ &= 2^{2n} \pi^{2n} (-1)^n \sum_{h \geq 0} \frac{(-2\pi i x)^h}{h!} \sum_{k \geq 1} (-1)^k k^{2n-1+h} = 2^{2n} \pi^{2n} (-1)^n \sum_{h \geq 0} \frac{(2\pi x)^{2h} (-1)^h}{(2h)!} \sum_{k \geq 1} (-1)^k k^{2n-1+2h}; \end{aligned}$$

ricordiamo che $\sum_{k \geq 1} (-1)^k k^{2n+2h} = 0$;

tenendo presente la (A.1.1.05), troviamo:

$$2^{2n} \pi^{2n} (-1)^n \sum_{k \geq 1} (-1)^k k^{2n-1} e^{-2\pi i x k} = 2^{2n} \pi^{2n} (-1)^{n-1} \sum_{h \geq 0} \frac{(2\pi x)^{2h} (-1)^h}{(2h)!} \frac{(2^{2n+2h} - 1) B_{2n+2h}}{2n+2h}, \text{ e quindi:}$$

$$(2n-1)! \sum_{k \geq 0} \left[\frac{1}{(\frac{1}{2} - x + k)^{2n}} + \frac{1}{(\frac{1}{2} + x + k)^{2n}} \right] = 2^{2n} \pi^{2n} \sum_{h \geq 0} \frac{(2\pi x)^{2h}}{(2h)!} \frac{(2^{2n+2h} - 1) |B_{2n+2h}|}{2n+2h}, \quad (\text{A.1.5.05})$$

Dalla (A.1.5.05), ricaviamo:

$$\begin{aligned} \sum_{k \geq 0} \left[\frac{1}{(\frac{1}{2} - x + k)^{2n}} + \frac{1}{(\frac{1}{2} + x + k)^{2n}} \right] &= \frac{2^{2n}}{(2n-1)!} \pi^{2n} \sum_{h \geq 0} \frac{(2\pi x)^{2h}}{(2h)!} \frac{(2^{2n+2h} - 1) |B_{2n+2h}|}{2n+2h} = \\ &= \frac{\pi^{2n}}{(2n-1)!} \sum_{h=0}^n a_h [Tan(\pi x)]^{2h} \end{aligned} \quad (\text{A.1.5.06})$$

che, per $x = 0$, fornisce la (A.1.5.04), e per $x = 1/4$, fornisce:

$$\begin{aligned} \sum_{k \geq 0} \left[\frac{1}{(1+4k)^{2n}} + \frac{1}{(3+4k)^{2n}} \right] &= \frac{\pi^{2n}}{2^{2n} (2n-1)!} \sum_{h \geq 0} \frac{(\pi/2)^{2h}}{(2h)!} \frac{(2^{2n+2h} - 1) |B_{2n+2h}|}{2n+2h} = \\ &= \frac{\pi^{2n}}{2(2n)!} (2^{2n} - 1) |B_{2n}| \end{aligned} \quad (\text{A.1.5.07})$$

A.2.0- Derivata di ordine (2n) della funzione P(x) = Tan(x)

Derivando la (A.0.0.04), (2n) volte, rispetto ad x, troviamo:

$$\begin{aligned} \int_0^{\infty} \frac{u^{2n} \text{Sinh}(ux)}{\text{Sinh}(u/2)} du &= \pi [\text{Tan}(\pi x)]^{(2n)} = \\ &= \pi^{2n+1} \sum_{h=0}^n b_h [\text{Tan}(\pi x)]^{2h+1} = \frac{2\pi}{i} \sum_{k \geq 1} (-1)^k (-2\pi i)^{2n} k^{2n} e^{-2\pi i x k} \end{aligned} \quad (\text{A.2.0.01})$$

Non è difficile verificare la relazione

$$\pi [\text{Tan}(\pi x)]^{(2n)} = \pi^{2n+1} \sum_{h=0}^n b_h [\text{Tan}(\pi x)]^{2h+1},$$

e la relazione
$$[\text{Tan}(x)]^{(2n)} = \sum_{h=0}^n b_h [\text{Tan}(x)]^{2h+1}$$

A.2.1- Determinazione del coefficiente b_0 del polinomio rappresentato

dallo sviluppo della funzione $[\text{Tan}(x)]^{(2n)} = \sum_{h=0}^n b_h [\text{Tan}(x)]^{2h+1}$

Operando sulla (A.2.0.01), abbiamo:

$$\begin{aligned} \int_0^{\infty} \frac{u^{2n} \text{Sinh}(ux)}{\text{Sinh}(u/2)} du &= \pi [\text{Tan}(\pi x)]^{2n} = \\ &= \pi^{2n+1} \sum_{h=0}^n b_h [\text{Tan}(\pi x)]^{2h+1} = \frac{(2\pi)^{2n+1} (-1)^n}{i} \sum_{k \geq 1} (-1)^k k^{2n} e^{-2\pi i x k} \end{aligned} \quad (\text{A.2.1.01})$$

Ponendo, nella (A.2.1.01), $\text{Tan}(\pi x) = t$, da cui $\pi x = \text{ArcTan}(t)$, otteniamo:

$$\int_0^{\infty} \frac{u^{2n} \text{Sinh}\left[\frac{u}{\pi} \text{ArcTan}(t)\right]}{\text{Sinh}(u/2)} du = \pi^{2n+1} \sum_{h=0}^n b_h t^{2h+1} \quad (\text{A.2.1.02})$$

Derivando la precedente (A.2.1.02), rispetto a (t), e ponendo dopo, $t = 0$, troviamo:

$$\lim_{t \rightarrow 0} D_t \int_0^{\infty} \frac{u^{2n} \text{Sinh}\left[\frac{u}{\pi} \text{ArcTan}(t)\right]}{\text{Sinh}(u/2)} du = \frac{1}{\pi} \int_0^{\infty} \frac{u^{2n+1}}{\text{Sinh}(u/2)} du = \pi^{2n+1} b_0, \text{ da cui:}$$

$$\frac{2}{\pi} \int_0^{\infty} u^{2n+1} e^{-u/2} \sum_{k \geq 0} e^{-uk} du = \frac{2}{\pi} \sum_{k \geq 0} \frac{(2n+1)!}{\left(\frac{1}{2} + k\right)^{2n+2}} = \frac{2^{2n+3} (2n+1)!}{\pi} \sum_{k \geq 0} \frac{1}{(1+2k)^{2n+2}};$$

applicando la (A.1.1.07), troviamo: $\sum_{k \geq 0} \frac{1}{(1+2k)^{2n+2}} = [1 - 2^{-(2n+2)}] \zeta(2n+2)$, e quindi:

$$\pi^{2n+1} b_0 = \frac{2^{2n+3} (2n+1)!}{\pi} [1 - 2^{-(2n+2)}] \zeta(2n+2) = \frac{2(2n+1)!}{\pi} (2^{2n+2} - 1) \zeta(2n+2);$$

ma: $\zeta(2n+2) = \frac{2^{2n+1} \pi^{2n+2}}{(2n+2)!} |B_{2n+2}|$, e pertanto:

$$b_0 = \frac{2^{2n+1}}{n+1} (2^{2n+2} - 1) |B_{2n+2}| \quad (\text{A.2.1.03})$$

La relazione (A.2.1.03) è stata verificata con un programma di matematica.

Riportiamo i primi valori di b_0 per n variabile da 1 a 10.

$$n = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10$$

$$b_0 = 2, 16, 272, 7936, 353792, 22368256, 1903757312, 209865342476, \\ 29088885112832, 4951498053124096$$

A.2.2 -Determinazione dell'espressione della Somma $\sum_{h=0}^n b_h$, in funzione di n .

Utilizzando la relazione (A.2.1.01), e ponendo, $x = 1/4$, troviamo:

$$\int_0^\infty \frac{u^{2n} \text{Sinh}(u/4)}{\text{Sinh}(u/2)} du = \pi^{2n+1} \sum_{h=0}^n b_h = \frac{(2\pi)^{2n+1} (-1)^n}{i} \sum_{k \geq 1} (-1)^k k^{2n} (-i)^k, \quad (\text{A.2.2.01})$$

da cui:

$$\sum_{h=0}^n b_h = \frac{2^{2n+1} (-1)^n}{i} \sum_{k \geq 1} (i)^k k^{2n}, \text{ ma } \sum_{k \geq 1} (i)^k k^{2n} = \sum_{k \geq 1} (-1)^k (2k)^{2n} + \sum_{k \geq 1} (i)^{2k-1} (2k-1)^{2n};$$

$$\text{quindi: } \sum_{h=0}^n b_h = \frac{2^{2n+1} (-1)^n}{i} \left[\sum_{k \geq 1} (-1)^k (2k)^{2n} + \sum_{k \geq 1} (i)^{2k-1} (2k-1)^{2n} \right].$$

Dalla precedente, uguagliando le parti reali, ricaviamo:

$$\sum_{h=0}^n b_h = \frac{2^{2n+1} (-1)^n}{i} \sum_{k \geq 1} (i)^{2k-1} (2k-1)^{2n} = 2^{2n+1} (-1)^{n-1} \sum_{k \geq 1} (-1)^k (2k-1)^{2n}; \text{ inoltre,}$$

$$\sum_{k \geq 1} (-1)^k (2k-1)^{2n} = \sum_{k \geq 1} (-1)^k (1-2k)^{2n} = \sum_{k \geq 1} (-1)^k \sum_{h \geq 0} \binom{2n}{h} (-2k)^h = \\ = \sum_{k \geq 1} (-1)^k + \sum_{h \geq 1} \binom{2n}{h} (-2)^h \sum_{k \geq 1} (-1)^k k^h;$$

$$\text{poiché } \sum_{k \geq 1} (-1)^k = -\frac{1}{2}, \text{ (Ved. [4], pag.2)} \quad (\text{A.2.2.02})$$

e ricordando che: $\zeta(-2h) = 0$, ($h = 1, 2, 3, \dots, n$), otteniamo:

$$\sum_{k \geq 1} (-1)^k (2k-1)^{2n} = -\frac{1}{2} + \sum_{h \geq 1} \binom{2n}{2h-1} (-2)^{2h-1} \sum_{k \geq 1} (-1)^k k^{2h-1}$$

Ricordando la (A.1.1.05), abbiamo :

$$\sum_{k \geq 1} (-1)^k (2k-1)^{2n} = -\frac{1}{2} - \sum_{h \geq 1} \binom{2n}{2h-1} (-2)^{2h-1} (2^{2h} - 1) \frac{B_{2h}}{2h};$$

osserviamo che: $\binom{2n}{2h-1} \frac{1}{2h} = \frac{1}{2n+1} \binom{2n+1}{2h}$; pertanto:

$$\sum_{k \geq 1} (-1)^k (2k-1)^{2n} = -\frac{1}{2} + \frac{1}{2(2n+1)} \sum_{h \geq 1} \binom{2n+1}{2h} (-2)^{2h} (2^{2h} - 1) B_{2h}, \quad (\text{A.2.2.03})$$

e quindi:

$$\sum_{h=0}^n b_h = 2^{2n} (-1)^{n-1} \left[\frac{1}{2n+1} \sum_{h=1}^n \binom{2n+1}{2h} 2^{2h} (2^{2h} - 1) B_{2h} - 1 \right] \quad (\text{A.2.2.04})$$

La relazione (A.2.2.04) è stata verificata con un programma di matematica.

Riportiamo i primi valori di $\sum_{h=0}^n b_h$ per n variabile da 1 a 10.

$$n = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10$$

$$\sum_{h=0}^n b_h = 4, 80, 3904, 354560, 51733504, 11070525440, 3266330312704,$$

$$1270842139934720, 630424777638805504, 388362339077351014400$$

A.2.3 - Determinazione delle espressioni che definiscono, in funzione di n,

i coefficienti $b_1, b_2, b_3, \dots, b_n$ dei termini del polinomio $\sum_{h=0}^n b_h [Tan(x)]^{2h+1}$

Ponendo, nella (A.2.1.01), $Tan(\pi x) = t$, da cui $\pi x = ArcTan(t) = \frac{1}{2i} \ln \frac{1+it}{1-it}$, otteniamo:

$$\begin{aligned} \pi^{2n+1} \sum_{h=0}^n b_h t^{2h+1} &= \frac{(2\pi)^{2n+1} (-1)^n}{i} \sum_{k \geq 1} (-1)^k k^{2n} e^{-k \ln \frac{1+it}{1-it}} = \\ &= \frac{(2\pi)^{2n+1} (-1)^n}{i} \sum_{k \geq 1} (-1)^k k^{2n} \left(\frac{1-it}{1+it} \right)^k; \end{aligned} \quad (A.2.3.01)$$

derivando la (A.2.3.01), (2h+1) volte, rispetto a t, e ponendo dopo, t = 0, abbiamo:

$$\begin{aligned} \pi^{2n+1} b_h (2h+1)! &= \left[\frac{(2\pi)^{2n+1} (-1)^n}{i} \sum_{k \geq 1} (-1)^k k^{2n} \left(\frac{1-it}{1+it} \right)^k \right]_{t=0}^{(2h+1)}; \\ \text{ora, } [(1-it)^k (1+it)^{-k}]_{t=0}^{(2h+1)} &= \lim_{t \rightarrow 0} \sum_{j=0}^{2h+1} \binom{2h+1}{j} [(1-it)^k]^{(2h+1-j)} [(1+it)^{-k}]^{(j)} = \\ &= \sum_{j=0}^{2h+1} \binom{2h+1}{j} \frac{\Gamma(k+1)(-i)^{2h+1-j}}{\Gamma(k+1-2h-1+j)} \frac{\Gamma(k+j)(-i)^j}{\Gamma(k)} = \\ &= \sum_{j=0}^{2h+1} \binom{2h+1}{j} (-1)^h (-i)^k \frac{\Gamma(k+j)}{\Gamma(k+j-2h)} = \\ &= \sum_{j=0}^{2h+1} \binom{2h+1}{j} (-1)^h (-i)^k (k+j-1)(k+j-2)\dots(k+j-2h) = \\ &= \sum_{j=0}^{2h+1} \binom{2h+1}{j} (-1)^h (-i)^k \sum_{u=0}^{2h} s(2h, u) (k+j-1)^u = \\ &= \sum_{j=0}^{2h+1} \binom{2h+1}{j} (-1)^h (-i)^k \sum_{u=0}^{2h} s(2h, u) \sum_{p=0}^u \binom{u}{p} k^p (j-1)^{u-p}; \text{ pertanto:} \end{aligned}$$

$$\begin{aligned} b_h (2h+1)! &= \\ &= \frac{2^{2n+1} (-1)^n}{i} \sum_{k \geq 1} (-1)^k k^{2n} \sum_{j=0}^{2h+1} \binom{2h+1}{j} (-1)^h (-i)^k \sum_{u=0}^{2h} s(2h, u) \sum_{p=0}^u \binom{u}{p} k^p (j-1)^{u-p} = \end{aligned}$$

$$\begin{aligned}
&= 2^{2n+1} (-1)^{n-1} (-1)^h \sum_{j=0}^{2h+1} \binom{2h+1}{j} \sum_{u=0}^{2h} s(2h, u) \sum_{p=0}^u \binom{u}{p} (j-1)^{u-p} \sum_{k \geq 1} (-1)^k k^{2n+p+1} = \\
&= 2^{2n+1} (-1)^{n-1} (-1)^h \sum_{j=0}^{2h+1} \binom{2h+1}{j} \sum_{u=0}^{2h} s(2h, u) \sum_{p=0}^{\lfloor u/2 \rfloor} \binom{u}{2p} (j-1)^{u-2p} \sum_{k \geq 1} (-1)^k k^{2n+2p+1} = \\
&= 2^{2n+1} (-1)^{n+h} \sum_{j=0}^{2h+1} \binom{2h+1}{j} \sum_{u=0}^{2h} s(2h, u) \sum_{p=0}^{\lfloor u/2 \rfloor} \binom{u}{2p} (j-1)^{u-2p} (2^{2n+2p+2} - 1) \frac{B_{2n+2p+2}}{2n+2p+2}. \quad (\text{A.2.3.02})
\end{aligned}$$

Anche qui, osserviamo che nello sviluppo della relazione precedente (A.2.3.02), al variare di j , u , p , si presentano valori indeterminati (0^0).

Per eliminare detti inconvenienti calcoliamo l'espressione precedente per $j = 0$, per $j = 1$, e per $j > 1$. Così operando, troviamo:

$$\begin{aligned}
1) \text{ per } j = 0, & \sum_{j=0}^{2h+1} \binom{2h+1}{j} (-1)^h (-i) k (k+j-1)(k+j-2) \dots (k+j-2h) = \\
&= (-1)^h (-i) k (k-1)(k-2) \dots (k-2h) = (-1)^h (-i) \sum_{u=0}^{2h+1} s(2h+1, u) k^u \\
[b_h(2h+1)!]_{j=0} &= \frac{2^{2n+1} (-1)^n}{i} \sum_{k \geq 1} (-1)^k k^{2n} (-1)^h (-i) \sum_{u=0}^{2h+1} s(2h+1, u) k^u = \\
&= 2^{2n+1} (-1)^{n-1} (-1)^h \sum_{u=0}^{2h+1} s(2h+1, u) \sum_{k \geq 1} (-1)^k k^{2n+u} = \\
&= 2^{2n+1} (-1)^{n-1} (-1)^h \sum_{u=0}^h s(2h+1, 2u+1) \sum_{k \geq 1} (-1)^k k^{2n+2u+1} = \\
&= 2^{2n+1} (-1)^n (-1)^h \sum_{u=0}^h s(2h+1, 2u+1) (2^{2n+2u+2} - 1) \frac{B_{2n+2u+2}}{2n+2u+2} \\
2) \text{ per } j = 1, & \sum_{j=0}^{2h+1} \binom{2h+1}{j} (-1)^h (-i) k (k+j-1)(k+j-2) \dots (k+j-2h) = \\
&= (2h+1) (-1)^h (-i) k (k)(k-1) \dots (k+1-2h) = (2h+1) (-1)^h (-i) k \sum_{u=0}^{2h} s(2h, u) k^u \\
[b_h(2h+1)!]_{j=1} &= \frac{2^{2n+1} (-1)^n}{i} \sum_{k \geq 1} (-1)^k k^{2n} (2h+1) (-1)^h (-i) k \sum_{u=0}^{2h} s(2h, u) k^u = \\
&= 2^{2n+1} (-1)^{n-1} (2h+1) (-1)^h \sum_{u=0}^{2h} s(2h, u) \sum_{k \geq 1} (-1)^k k^{2n+1+u} = \\
&= 2^{2n+1} (-1)^{n-1} (2h+1) (-1)^h \sum_{u=0}^h s(2h, 2u) \sum_{k \geq 1} (-1)^k k^{2n+1+2u} = \\
&= 2^{2n+1} (-1)^n (2h+1) (-1)^h \sum_{u=0}^h s(2h, 2u) (2^{2n+2u+2} - 1) \frac{B_{2n+2u+2}}{2n+2u+2};
\end{aligned}$$

Pertanto, ricaviamo:

$$\begin{aligned}
b_h(2h+1)! &= 2^{2n+1}(-1)^n(-1)^h \sum_{u=0}^h s(2h+1,2u+1)(2^{2n+2u+2}-1) \frac{B_{2n+2u+2}}{2n+2u+2} + \\
&+ 2^{2n+1}(-1)^n(2h+1)(-1)^h \sum_{u=0}^h s(2h,2u)(2^{2n+2u+2}-1) \frac{B_{2n+2u+2}}{2n+2u+2} + \\
&+ 2^{2n+1}(-1)^{n+h} \sum_{j=2}^{2h+1} \binom{2h+1}{j} \sum_{u=0}^{2h} s(2h,u) \sum_{p=0}^{\lfloor u/2 \rfloor} \binom{u}{2p} (j-1)^{u-2p} (2^{2n+2p+2}-1) \frac{B_{2n+2p+2}}{2n+2p+2}, \\
\text{da cui: } b_h &= \frac{2^{2n+1}}{(2h+1)!} (-1)^n (-1)^h \left[\sum_{u=0}^h s(2h+1,2u+1)(2^{2n+2u+2}-1) \frac{B_{2n+2u+2}}{2n+2u+2} + \right. \\
&+ (2h+1) \sum_{u=0}^h s(2h,2u)(2^{2n+2u+2}-1) \frac{B_{2n+2u+2}}{2n+2u+2} + \\
&+ \left. \sum_{j=2}^{2h+1} \binom{2h+1}{j} \sum_{u=0}^{2h} s(2h,u) \sum_{p=0}^{\lfloor u/2 \rfloor} \binom{u}{2p} (j-1)^{u-2p} (2^{2n+2p+2}-1) \frac{B_{2n+2p+2}}{2n+2p+2} \right] \tag{A.2.3.03}
\end{aligned}$$

Per $h = 0$, ritroviamo la formula (A.2.1.03): $b_0 = \frac{2^{2n+1}}{n+1} (2^{2n+2}-1) |B_{2n+2}|$

Per $h = n$, ricaviamo:

$$\begin{aligned}
b_n &= \frac{2^{2n+1}}{(2n+1)!} \left[\sum_{u=0}^n s(2n+1,2u+1)(2^{2n+2u+2}-1) \frac{B_{2n+2u+2}}{2n+2u+2} + \right. \\
&+ (2n+1) \sum_{u=0}^n s(2n,2u)(2^{2n+2u+2}-1) \frac{B_{2n+2u+2}}{2n+2u+2} \\
&+ \left. \sum_{j=2}^{2n+1} \binom{2n+1}{j} \sum_{u=0}^{2n} s(2n,u) \sum_{p=0}^{\lfloor u/2 \rfloor} \binom{u}{2p} (j-1)^{u-2p} (2^{2n+2p+2}-1) \frac{B_{2n+2p+2}}{2n+2p+2} \right] = (2n)! \tag{A.2.3.04}
\end{aligned}$$

Le relazioni (A.2.3.03) e (a.2.3.04) sono state verificate con un programma di matematica.

E' straordinario verificare che un'espressione molto complessa, come quella di cui al 2° membro della (A.2.3.04), risulta uguale a $(2n)!$

A.2.4 – Consideriamo la relazione (A.2.1.01),

$$\begin{aligned}
\int_0^\infty \frac{u^{2n} \text{Sinh}(ux)}{\text{Sinh}(u/2)} du &= \pi [\text{Tan}(\pi x)]^{2n} = \\
&= \pi^{2n+1} \sum_{h=0}^n b_h [\text{Tan}(\pi x)]^{2h+1} = \frac{(2\pi)^{2n+1} (-1)^n}{i} \sum_{k \geq 0} (-1)^k (1+k)^{2n} e^{-2\pi i x(1+k)};
\end{aligned}$$

ponendo, in detta relazione, $x = 0$, otteniamo:

$$\sum_{k \geq 0} (-1)^k (1+k)^{2n} = 0; \quad (\text{A.2.4.01})$$

$$\sum_{k \geq 0} (-1)^k (1+k)^{2n} = 1 + \sum_{k \geq 1} (-1)^k \sum_{h \geq 0} \binom{2n}{h} k^h = 1 + \sum_{k \geq 1} (-1)^k + \sum_{h \geq 1} \binom{2n}{h} \sum_{k \geq 1} (-1)^k k^h = 0;$$

per la (A.2.2.02), $\sum_{k \geq 1} (-1)^k = -\frac{1}{2}$; e $\zeta(-2h) = 0$, ($h = 1, 2, 3, \dots$); pertanto:

$$\sum_{k \geq 0} (-1)^k (1+k)^{2n} = \frac{1}{2} + \sum_{h \geq 1} \binom{2n}{2h-1} \sum_{k \geq 1} (-1)^k k^{2h-1};$$

per la (A.1.1.05), otteniamo:

$$\sum_{k \geq 0} (-1)^k (1+k)^{2n} = \frac{1}{2} + \sum_{h \geq 1} \binom{2n}{2h-1} (2^{2h} - 1) \left(-\frac{B_{2h}}{2h}\right) = 0$$

ricordando che: $\binom{2n}{2h-1} \frac{1}{2h} = \frac{1}{2n+1} \binom{2n+1}{2h}$, troviamo:

$$\frac{1}{2} - \frac{1}{2n+1} \sum_{h \geq 1} \binom{2n+1}{2h} (2^{2h} - 1) B_{2h} = 0, \text{ da cui:}$$

$$\frac{2}{2n+1} \sum_{h=1}^n \binom{2n+1}{2h} (2^{2h} - 1) B_{2h} = 1 \quad (\text{A.2.4.02})$$

La formula (A.2.4.02) è stata verificata con un programma di matematica.

A.2.5 - Ponendo nella relazione (A.2.1.01), $x = 1/4$, troviamo:

$$\int_0^{\infty} \frac{u^{2n} \text{Sinh}(u/4)}{\text{Sinh}(u/2)} du = \pi [\text{Tan}(\pi x)]_{x=1/4}^{(2n)} = \frac{-(2\pi)^{2n+1} (-1)^n}{i} \sum_{k \geq 0} (-1)^k (1+k)^{2n} e^{-2\pi i(1+k)/4};$$

$$\begin{aligned} \int_0^{\infty} \frac{u^{2n} \text{Sinh}(u/4)}{\text{Sinh}(u/2)} du &= \int_0^{\infty} \frac{u^{2n}}{2 \text{Cosh}(u/4)} du = \int_0^{\infty} u^{2n} e^{-u/4} \sum_{k \geq 0} (-1)^k e^{-uk/2} du = \\ &= \sum_{k \geq 0} (-1)^k \frac{(2n)!}{\left(\frac{1}{4} + \frac{k}{2}\right)^{2n+1}} = 4^{2n+1} (2n)! \sum_{k \geq 0} \frac{(-1)^k}{(1+2k)^{2n+1}}; \\ &= \frac{-(2\pi)^{2n+1} (-1)^n}{i} \sum_{k \geq 0} (-1)^k (1+k)^{2n} e^{-2\pi i(1+k)/4} = 2^{2n+1} \pi^{2n+1} (-1)^n \sum_{k \geq 0} (i)^k (1+k)^{2n} = \\ &= 2^{2n+1} \pi^{2n+1} (-1)^n \left[\sum_{k \geq 0} (-1)^k (1+2k)^{2n} + \sum_{k \geq 1} (i)^{2k-1} (2k)^{2n} \right] = 4^{2n+1} (2n)! \sum_{k \geq 0} \frac{(-1)^k}{(1+2k)^{2n+1}}; \end{aligned}$$

uguagliando le parti reali, troviamo:

$$4^{2n+1} (2n)! \sum_{k \geq 0} \frac{(-1)^k}{(1+2k)^{2n+1}} = \pi^{2n+1} \sum_{h=0}^n b_h = 2^{2n+1} \pi^{2n+1} (-1)^n \sum_{k \geq 0} (-1)^k (1+2k)^{2n};$$

$$\sum_{k \geq 0} (-1)^k (1+2k)^{2n} = 1 + \sum_{k \geq 1} (-1)^k \sum_{h \geq 0} \binom{2n}{h} (2k)^h = 1 + \sum_{k \geq 1} (-1)^k + \sum_{h \geq 1} \binom{2n}{h} 2^h \sum_{k \geq 1} (-1)^k k^h;$$

per la (A.2.2.02), e ricordando che $\zeta(-2h) = 0$, ($h = 1, 2, 3, \dots$), otteniamo:

$$\sum_{k \geq 0} (-1)^k (1+2k)^{2n} = \frac{1}{2} + \sum_{h \geq 1} \binom{2n}{2h-1} 2^{2h-1} \sum_{k \geq 1} (-1)^k k^{2h-1} =$$

$$= \frac{1}{2} - \frac{1}{2(2n+1)} \sum_{h \geq 1} \binom{2n+1}{2h} 2^{2h} (2^{2h} - 1) B_{2h}; \quad (\text{A.2.5.01})$$

quindi: $4^{2n+1} (2n)! \sum_{k \geq 0} \frac{(-1)^k}{(1+2k)^{2n+1}} = \pi^{2n+1} \sum_{h=0}^n b_h =$

$$= 2^{2n+1} \pi^{2n+1} (-1)^{n-1} \left[-\frac{1}{2} + \frac{1}{2(2n+1)} \sum_{h \geq 1} \binom{2n+1}{2h} 2^{2h} (2^{2h} - 1) B_{2h} \right]; \text{ cioè:}$$

$$\sum_{k \geq 0} \frac{(-1)^k}{(1+2k)^{2n+1}} = \frac{\pi^{2n+1} (-1)^{n-1}}{2^{2n+2} (2n+1)!} \left[\sum_{h=1}^n \binom{2n+1}{2h} 2^{2h} (2^{2h} - 1) B_{2h} - (2n+1) \right] \quad (\text{A.2.5.02})$$

Osserviamo che:

$$\sum_{k \geq 0} \frac{(-1)^k}{(1+2k)^{2n+1}} = \sum_{k \geq 0} \left[\frac{1}{(1+4k)^{2n+1}} - \frac{1}{(3+4k)^{2n+1}} \right]$$

La formula (A.2.5.02) è stata verificata con un programma di matematica.

A.2.6. – Consideriamo il caso generale fornito dalla relazione (A.2.1.01)

$$\int_0^\infty \frac{u^{2n} \text{Sinh}(ux)}{\text{Sinh}(u/2)} du = \pi [\text{Tan}(\pi x)]^{(2n)} = \frac{-(2\pi)^{2n+1} (-1)^n}{i} \sum_{k \geq 0} (-1)^k (1+k)^{2n} e^{-2\pi i x(1+k)}$$

Sviluppando, abbiamo:

$$\int_0^\infty \frac{u^{2n} \text{Sinh}(ux)}{\text{Sinh}(u/2)} du = \int_0^\infty u^{2n} (e^{ux} - e^{-ux}) e^{-u/2} \sum_{k \geq 0} e^{-uk} du =$$

$$= (2n)! \sum_{k \geq 0} \left[\frac{1}{\left(\frac{1}{2} - x + k\right)^{2n+1}} - \frac{1}{\left(\frac{1}{2} + x + k\right)^{2n+1}} \right] = \pi^{2n+1} \sum_{h=0}^n b_h [\text{Tan}(x)]^{2h+1}; \quad (\text{A.2.6.01})$$

Inoltre:

$$\frac{(2\pi)^{2n+1} (-1)^n}{i} \sum_{k \geq 1} (-1)^k k^{2n} e^{-2\pi i x k} =$$

$$= \frac{(2\pi)^{2n+1} (-1)^n}{i} \sum_{k \geq 1} (-1)^k k^{2n} \sum_{h \geq 0} \frac{(-2\pi i x k)^h}{h!}; \quad (\text{A.2.6.02})$$

per x reale, l'ultimo membro della (A.2.6.02) è reale solamente per h dispari; quindi:

$$\pi^{2n+1} \sum_{h=0}^n b_h [\text{Tan}(x)]^{2h+1} = \frac{(2\pi)^{2n+1} (-1)^n}{i} \sum_{k \geq 1} (-1)^k k^{2n} \sum_{h \geq 1} \frac{(-2\pi i x)^{2h-1} k^{2h-1}}{(2h-1)!} =$$

$$= (2\pi)^{2n+1} (-1)^n \sum_{h \geq 1} \frac{(2\pi x)^{2h-1} (-1)^h}{(2h-1)!} \sum_{k \geq 0} (-1)^k (k)^{2n+2h-1} =$$

$$= (2\pi)^{2n+1} (-1)^{n-1} \sum_{h \geq 1} \frac{(2\pi x)^{2h-1} (-1)^h}{(2h-1)!} \frac{(2^{2n+2h} - 1) B_{2n+2h}}{2n+2h}$$

Pertanto: $(2n)! \sum_{k \geq 0} \left[\frac{1}{\left(\frac{1}{2} - x + k\right)^{2n+1}} - \frac{1}{\left(\frac{1}{2} + x + k\right)^{2n+1}} \right] = \pi^{2n+1} \sum_{h=0}^n b_h [\text{Tan}(x)]^{2h+1} =$

$$= (2\pi)^{2n+1} (-1)^{n-1} \sum_{h \geq 1} \frac{(2\pi x)^{2h-1} (-1)^h (2^{2n+2h} - 1) B_{2n+2h}}{(2h-1)! (2n+2h)} \quad (\text{A.2.6.03})$$

Per, $x = 1/4$, troviamo: $(2n)! 4^{2n+1} \sum_{k \geq 0} \left[\frac{1}{(1+4k)^{2n+1}} - \frac{1}{(3+4k)^{2n+1}} \right] = \pi^{2n+1} \sum_{h=0}^n b_h =$

$$= (2\pi)^{2n+1} \sum_{h \geq 1} \frac{(\pi/2)^{2h-1} (2^{2n+2h} - 1) |B_{2n+2h}|}{(2h-1)! (2n+2h)} \quad (\text{A.2.6.04})$$

da cui: $\sum_{k \geq 0} \left[\frac{1}{(1+4k)^{2n+1}} - \frac{1}{(3+4k)^{2n+1}} \right] = \sum_{k \geq 0} \frac{(-1)^k}{(1+2k)^{2n+1}} = \frac{\pi^{2n+1}}{(2n)! 4^{2n+1}} \sum_{h=0}^n b_h =$

$$= \frac{(2\pi)^{2n+1}}{(2n)! 4^{2n+1}} \sum_{h \geq 1} \frac{(\pi/2)^{2h-1} (2^{2n+2h} - 1) |B_{2n+2h}|}{(2h-1)! (2n+2h)} \quad (\text{A.2.6.05})$$

Dalla (A.2.6.04), otteniamo:

$$\frac{2^{2n+1}}{(2n)! 4^{2n+1}} \sum_{h \geq 1} \frac{(\pi/2)^{2h-1} (2^{2n+2h} - 1) |B_{2n+2h}|}{(2h-1)! (2n+2h)} = \sum_{h=0}^n b_h \quad (\text{A.2.6.06})$$

B.0.0 – Caso B - $C(x) = \text{Sec}(x)$

Utilizzando la seguente relazione dei complementi:

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\text{Sin}(\pi z)}, \quad 0 < z < 1$$

e ponendo, $z = \frac{1}{2} - x$, (x , reale), otteniamo:

$$\Gamma\left(x + \frac{1}{2}\right)\Gamma\left(\frac{1}{2} - x\right) = \frac{\pi}{\text{Cos}(\pi x)}, \quad |x| < \frac{1}{2} \quad (\text{B.0.0.01})$$

Ora, $\Gamma\left(x + \frac{1}{2}\right)\Gamma\left(\frac{1}{2} - x\right) = \int_0^1 t^{x-1+1/2} (1-t)^{-x-1+1/2} dt = \int_0^\infty t^{x-1+1/2} \frac{dt}{1+t} = (t = e^u) =$

$$= \int_{-\infty}^\infty \frac{e^{u(x-1+1/2)} e^u du}{1+e^u} = \int_0^\infty \frac{e^{u(x-1+1/2)} e^u du}{1+e^u} + \int_0^\infty \frac{e^{-u(x-1+1/2)} e^{-u} du}{1+e^{-u}} =$$

$$= \int_0^\infty \frac{e^{u(x+1/2)} du}{1+e^u} + \int_0^\infty \frac{e^{-u(x+1/2)} du}{1+e^{-u}} =$$

$$= \int_0^\infty \frac{e^{ux}}{e^{u/2} + e^{-u/2}} du + \int_0^\infty \frac{e^{-ux}}{e^{u/2} + e^{-u/2}} du =$$

$$= \int_0^\infty \frac{\text{Cosh}(ux)}{\text{Cosh}(u/2)} du = \pi \text{Sec}(\pi x) = \frac{\pi}{\text{Cos}(\pi x)}, \quad |x| < \frac{1}{2} \quad (\text{B.0.0.02})$$

B.1.1- Derivando, $(2n-1)$ volte, rispetto ad x , la relazione (B.0.0.02), otteniamo:

$$\int_0^{\infty} \frac{u^{2n-1} \text{Sinh}(ux)}{\text{Cosh}(u/2)} du = \left[\frac{\pi}{\text{Cos}(\pi x)} \right]^{(2n-1)}, \quad |x| < \frac{1}{2}, \quad (\text{B.1.1.01})$$

Dalla (B.1.1.01), passando al limite, per $x \rightarrow 0$, abbiamo:

$$\lim_{x \rightarrow 0} \int_0^{\infty} \frac{u^{2n-1} \text{Sinh}(ux)}{\text{Cosh}(u/2)} du = \left[\frac{\pi}{\text{Cos}(\pi x)} \right]_{x=0}^{(2n-1)} = 0; \text{ cioè:}$$

$$\left[\frac{\pi}{\text{Cos}(\pi x)} \right]_{x=0}^{(2n-1)} = 2\pi \left[\sum_{k \geq 0} (-1)^k e^{-i\pi x(1+2k)} \right]_{x=0}^{(2n-1)} = 2\pi \sum_{k \geq 0} (-1)^k (-i\pi)^{2n-1} (1+2k)^{2n-1} = 0,$$

$$\text{da cui: } \sum_{k \geq 0} (-1)^k (1+2k)^{2n-1} = 0; \quad (\text{B.1.1.02})$$

$$\text{ora, } \sum_{k \geq 0} (-1)^k (1+2k)^{2n-1} = 1 + \sum_{k \geq 1} (-1)^k (1+2k)^{2n-1} = 1 + \sum_{k \geq 1} (-1)^k \sum_{h=0}^{2n-1} \binom{2n-1}{h} (2k)^h =$$

$$= 1 + \sum_{k \geq 1} (-1)^k + \sum_{k \geq 1} (-1)^k \sum_{h=1}^{2n-1} \binom{2n-1}{h} (2k)^h = 1 - \frac{1}{2} + \sum_{h=1}^{2n-1} \binom{2n-1}{h} 2^h \sum_{k \geq 1} (-1)^k k^h =$$

$$= \frac{1}{2} + \sum_{h=1}^n \binom{2n-1}{2h-1} 2^{2h-1} \sum_{k \geq 1} (-1)^k k^{2h-1} = \frac{1}{2} - \frac{1}{2} \sum_{h=1}^n \binom{2n-1}{2h-1} 2^{2h} (2^{2h} - 1) \frac{B_{2h}}{2h} = 0, \text{ da cui:}$$

$$\frac{1}{2n} \sum_{h=1}^n \binom{2n}{2h} 2^{2h} (2^{2h} - 1) B_{2h} = 1 \quad (\text{B.1.1.03})$$

Nel procedimento per ricavare la (B.1.1.03) abbiamo utilizzato le seguenti note formule:

$$(\text{A.1.1.05}), (\text{A.2.2.02}), \zeta(1-2h) = -\frac{B_{2h}}{2h}, \zeta(-2h) = 0, \binom{2n-1}{2h-1} \frac{1}{2h} = \frac{1}{2n} \binom{2n}{2h}$$

La relazione (B.1.1.03) è stata verificata con un programma di matematica.

B.1.2- Dalla (B.1.1.01), passando al limite, per $x \rightarrow 1/4$, abbiamo:

$$\lim_{x \rightarrow 1/4} \int_0^{\infty} \frac{u^{2n-1} \text{Sinh}(ux)}{\text{Cosh}(u/2)} du = \left[\frac{\pi}{\text{Cos}(\pi x)} \right]_{x=1/4}^{(2n-1)} = 2\pi \left[\sum_{k \geq 0} (-1)^k e^{-i\pi x(1+2k)} \right]_{x=1/4}^{(2n-1)};$$

$$\text{ora, } \int_0^{\infty} \frac{u^{2n-1} \text{Sinh}(u/4)}{\text{Cosh}(u/2)} du = \int_0^{\infty} u^{2n-1} (e^{u/4} - e^{-u/4}) e^{-u/2} \sum_{k \geq 0} (-1)^k e^{-uk} du =$$

$$\int_0^{\infty} u^{2n-1} \sum_{k \geq 0} (-1)^k (e^{-u/4} - e^{-3u/4}) e^{-uk} du = (2n-1)! \sum_{k \geq 0} (-1)^k \left[\frac{1}{\left(\frac{1}{4} + k\right)^{2n}} - \frac{1}{\left(\frac{3}{4} + k\right)^{2n}} \right] =$$

$$= (2n-1)! 4^{2n} \sum_{k \geq 0} (-1)^k \left[\frac{1}{(1+4k)^{2n}} - \frac{1}{(3+4k)^{2n}} \right], \text{ mentre:}$$

$$2\pi \left[\sum_{k \geq 0} (-1)^k e^{-i\pi x(1+2k)} \right]_{x=1/4}^{(2n-1)} = 2\pi \sum_{k \geq 0} (-1)^k (-i\pi)^{2n-1} (1+2k)^{2n-1} e^{-i\pi/4} (-i)^k =$$

$$= 2\pi^{2n} \frac{(-1)^n}{-i} e^{-i\pi/4} \sum_{k \geq 0} (i)^k (1+2k)^{2n-1};$$

$$\sum_{k \geq 0} (i)^k (1+2k)^{2n-1} = \sum_{k \geq 0} (-1)^k (1+4k)^{2n-1} + \sum_{k \geq 1} (i)^{2k-1} (4k-1)^{2n-1} =$$

$$= \sum_{k \geq 0} (-1)^k (1+4k)^{2n-1} + i \sum_{k \geq 1} (-1)^k (1-4k)^{2n-1}; \text{ quindi:}$$

$$2\pi^{2n} \frac{(-1)^n}{-i} e^{-i\pi/4} \sum_{k \geq 0} (i)^k (1+2k)^{2n-1} = \pi^{2n} (-1)^n \sqrt{2} (1+i) \sum_{k \geq 0} (i)^k (1+2k)^{2n-1} =$$

$$= \pi^{2n} (-1)^n \sqrt{2} (1+i) \left[\sum_{k \geq 0} (-1)^k (1+4k)^{2n-1} + i \sum_{k \geq 1} (-1)^k (1-4k)^{2n-1} \right]$$

Prendendo la parte reale della precedente espressione, abbiamo:

$$\pi^{2n} (-1)^n \sqrt{2} \left[\sum_{k \geq 0} (-1)^k (1+4k)^{2n-1} - \sum_{k \geq 1} (-1)^k (1-4k)^{2n-1} \right] =$$

$$= \pi^{2n} (-1)^n \sqrt{2} \left[1 + \sum_{k \geq 1} (-1)^k \sum_{h=0}^{2n-1} \binom{2n-1}{h} (4k)^h - \sum_{k \geq 1} (-1)^k \sum_{h=0}^{2n-1} \binom{2n-1}{h} (-4k)^h \right] =$$

$$= \pi^{2n} (-1)^n \sqrt{2} \left[1 + \sum_{k \geq 1} (-1)^k \sum_{h=1}^n \binom{2n-1}{2h-1} (4k)^{2h-1} - \sum_{k \geq 1} (-1)^k \sum_{h=1}^n \binom{2n-1}{2h-1} (-4k)^{2h-1} \right] =$$

$$= \pi^{2n} (-1)^n \sqrt{2} \left[1 + \frac{1}{2} \sum_{h=1}^n \binom{2n-1}{2h-1} 4^{2h} \sum_{k \geq 1} (-1)^k k^{2h-1} \right] =$$

$$= \pi^{2n} (-1)^n \sqrt{2} \left[1 + \frac{1}{2} \sum_{h=1}^n \binom{2n-1}{2h-1} 4^{2h} (2^{2h} - 1) \left(-\frac{B_{2h}}{2h}\right) \right] =$$

$$= \pi^{2n} (-1)^{n-1} \sqrt{2} \left[\frac{1}{4n} \sum_{h=1}^n \binom{2n}{2h} 4^{2h} (2^{2h} - 1) B_{2h} - 1 \right]; \text{ pertanto:}$$

$$\begin{aligned}
& (2n-1)!4^{2n} \sum_{k \geq 0} (-1)^k \left[\frac{1}{(1+4k)^{2n}} - \frac{1}{(3+4k)^{2n}} \right] = \\
& = \pi^{2n} (-1)^{n-1} \sqrt{2} \left[\frac{1}{4n} \sum_{h=1}^n \binom{2n}{2h} 4^{2h} (2^{2h} - 1) B_{2h} - 1 \right], \text{ da cui:} \\
& \sum_{k \geq 0} (-1)^k \left[\frac{1}{(1+4k)^{2n}} - \frac{1}{(3+4k)^{2n}} \right] = \\
& = \pi^{2n} (-1)^{n-1} \sqrt{2} \frac{4^{-2n}}{2(2n)!} \left[\sum_{h=1}^n \binom{2n}{2h} 4^{2h} (2^{2h} - 1) B_{2h} - 4n \right]. \tag{B.1.2.01}
\end{aligned}$$

La relazione (B.1.2.01) è stata verificata con un programma di matematica.

B.1.3 – Riprendiamo la (B.1.1.01)

$$\int_0^{\infty} \frac{u^{2n-1} \text{Sinh}(ux)}{\text{Cosh}(u/2)} du = \left[\frac{\pi}{\text{Cos}(\pi x)} \right]^{(2n-1)}, \quad |x| < \frac{1}{2} \tag{B.1.3.01}$$

Operando, abbiamo:

$$\begin{aligned}
& \int_0^{\infty} \frac{u^{2n-1} \text{Sinh}(ux)}{\text{Cosh}(u/2)} du = \int_0^{\infty} u^{2n-1} (e^{ux} - e^{-ux}) \sum_{k \geq 0} e^{-u/2} (-1)^k e^{-uk} du = \\
& = (2n-1)! \sum_{k \geq 0} (-1)^k \left[\frac{1}{\left(\frac{1}{2} - x + k\right)^{2n}} - \frac{1}{\left(\frac{1}{2} + x + k\right)^{2n}} \right]; \tag{B.1.3.02}
\end{aligned}$$

$$\begin{aligned}
& \left[\frac{\pi}{\text{Cos}(\pi x)} \right]^{(2n-1)} = \left[\frac{\pi}{\text{Cos}(\pi x)} \right]^{(2n-2+1)} = \pi \left[\frac{\pi \text{Sin}(\pi x)}{[\text{Cos}(\pi x)]^2} \right]^{(2n-2)} = \\
& = \pi [\text{Sin}(\pi x) D \text{Tan}(\pi x)]^{(2n-2)} = \pi \sum_{k=0}^{2n-2} \binom{2n-2}{k} [\text{Sin}(\pi x)]^{(2n-2-k)} [\text{Tan}(\pi x)]^{(k+1)} = \\
& = \pi \sum_{k=0}^{n-1} \binom{2n-2}{2k} [\text{Sin}(\pi x)]^{(2n-2-2k)} [\text{Tan}(\pi x)]^{(2k+1)} + \pi \sum_{k=1}^n \binom{2n-2}{2k-1} [\text{Sin}(\pi x)]^{(2n-1-2k)} [\text{Tan}(\pi x)]^{(2k)}
\end{aligned}$$

$$\begin{aligned}
& \text{Osserviamo che: } [\text{Sin}(\pi x)]^{(2n-1-2k)} = \left[\frac{1}{2i} (e^{i\pi x} - e^{-i\pi x}) \right]^{(2n-1-2k)} = \\
& = \left[\frac{1}{2i} [(i\pi)^{2n-1-2k} e^{i\pi x} - (-i\pi)^{2n-1-2k} e^{-i\pi x}] \right] = \pi^{2n-1-2k} (-1)^{n-k-1} \left[\frac{1}{2} [e^{i\pi x} + e^{-i\pi x}] \right] = \\
& = \pi^{2n-1-2k} (-1)^{n-k-1} \text{Cos}(\pi x) \tag{B.1.3.03}
\end{aligned}$$

$$\begin{aligned}
[\text{Sin}(\pi x)]^{(2n-2k)} &= \left[\frac{1}{2i} (e^{i\pi x} - e^{-i\pi x}) \right]^{(2n-2k)} = \left[\frac{1}{2i} [(i\pi)^{2n-2k} e^{i\pi x} - (-i\pi)^{2n-2k} e^{-i\pi x}] \right] = \\
&= \pi^{2n-2k} (-1)^{n-k} \left[\frac{1}{2i} [e^{i\pi x} - e^{-i\pi x}] \right] = \pi^{2n-2k} (-1)^{n-k} \text{Sin}(\pi x)
\end{aligned} \tag{B.1.3.04}$$

Applicando la (A.1.1.01) e la (A.2.1.01), ricaviamo:

$$\begin{aligned}
(2n-1)! \sum_{k \geq 0} (-1)^k \left[\frac{1}{\left(\frac{1}{2} - x + k\right)^{2n}} - \frac{1}{\left(\frac{1}{2} + x + k\right)^{2n}} \right] = \\
\pi^{2n} (-1)^{n-1} \left\{ \sum_{k=0}^{n-1} \binom{2n-2}{2k} (-1)^k \text{Sin}(\pi x) \sum_{h=0}^{k+1} a_h \text{Tan}(\pi x) \right\}^{2h} + \\
+ \sum_{k=1}^n \binom{2n-2}{2k-1} (-1)^k \text{Cos}(\pi x) \sum_{h=0}^k b_h [\text{Tan}(\pi x)]^{2h+1} \} = \\
= \pi^{2n} (-1)^{n-1} \text{Sin}(\pi x) \left\{ \sum_{k=0}^{n-1} \binom{2n-2}{2k} (-1)^k \sum_{h=0}^{k+1} a_h \text{Tan}(\pi x) \right\}^{2h} + \\
+ \sum_{k=1}^{n-1} \binom{2n-2}{2k-1} (-1)^k \text{Cos}(\pi x) \sum_{h=0}^k b_h [\text{Tan}(\pi x)]^{2h} \}
\end{aligned} \tag{B.1.3.05}$$

I coefficienti a_h, b_h sono forniti, rispettivamente, dalle formule (A.1.3.05) e (A.2.3.03).

Per $x = 1/4$, otteniamo:

$$\begin{aligned}
(2n-1)! 4^{2n} \sum_{k \geq 0} (-1)^k \left[\frac{1}{(1+4k)^{2n}} - \frac{1}{(3+4k)^{2n}} \right] = \\
= \pi^{2n} (-1)^{n-1} \frac{\sqrt{2}}{2} \left\{ \sum_{k=0}^{n-1} \binom{2n-2}{2k} (-1)^k \sum_{h=0}^{k+1} a_h + \sum_{k=1}^n \binom{2n-2}{2k-1} (-1)^k \sum_{h=0}^k b_h \right\};
\end{aligned} \tag{B.1.3.06}$$

in virtù delle relazioni (A.1.2.01) e (A.2.2.04), troviamo:

$$\begin{aligned}
(2n-1)! 4^{2n} \sum_{k \geq 0} (-1)^k \left[\frac{1}{(1+4k)^{2n}} - \frac{1}{(3+4k)^{2n}} \right] = \\
= \pi^{2n} (-1)^{n-1} \frac{\sqrt{2}}{2} \left\{ \sum_{k=0}^{n-1} \binom{2n-2}{2k} (-1)^k 2^{4k+3} (2^{2k+2} - 1) \frac{|B_{2k+2}|}{2k+2} + \right. \\
\left. - \sum_{k=1}^n \binom{2n-2}{2k-1} 2^{2k} \left[\frac{1}{2k+1} \sum_{h=1}^k \binom{2k+1}{2h} \right] 2^{2h} (2^{2h} - 1) B_{2h} - 1 \right\}
\end{aligned} \tag{B.1.3.07}$$

B.2.2.- Derivando, (2n) volte, rispetto ad x, la (B.0.02), otteniamo:

$$\int_0^{\infty} \frac{u^{2n} \text{Cosh}(ux)}{\text{Cosh}(u/2)} du = \left[\frac{\pi}{\text{Cos}(\pi x)} \right]^{(2n)} = \quad |x| < \frac{1}{2}, \quad (\text{B.2.2.01})$$

Ponendo, nella (B.2.2.01), $x = 0$, troviamo:

$$\int_0^{\infty} \frac{u^{2n}}{\text{Cosh}(u/2)} du = \lim_{x \rightarrow 0} \left[\frac{\pi}{\text{Cos}(\pi x)} \right]^{(2n)} = 2\pi \sum_{k \geq 0} (-1)^k (-i\pi)^{2n} (1+2k)^{2n};$$

$$\int_0^{\infty} \frac{u^{2n}}{\text{Cosh}(u/2)} du = 2 \int_0^{\infty} u^{2n} e^{-u/2} \sum_{k \geq 0} (-1)^k e^{-uk} du = 2(2n)! \sum_{k \geq 0} (-1)^k \frac{1}{\left(\frac{1}{2} + k\right)^{2n+1}} =$$

$$= 2^{2n+2} (2n)! \sum_{k \geq 0} (-1)^k \frac{1}{(1+2k)^{2n+1}};$$

$$2\pi \sum_{k \geq 0} (-1)^k (-i\pi)^{2n} (1+2k)^{2n} = 2\pi^{2n+1} (-1)^n \sum_{k \geq 0} (-1)^k (1+2k)^{2n} =$$

$$= 2\pi^{2n+1} (-1)^n \left[\frac{1}{2} - \frac{1}{2n+1} \sum_{h \geq 1} \binom{2n+1}{2h} 2^{2h-1} (2^{2h} - 1) B_{2h} \right];$$

abbiamo utilizzato la (A.2.5.01), per cui:

$$\sum_{k \geq 0} \frac{(-1)^k}{(1+2k)^{2n+1}} = \frac{\pi^{2n+1} (-1)^{n-1}}{2^{2n+2} (2n+1)!} \left[\sum_{h \geq 1} \binom{2n+1}{2h} 2^{2h} (2^{2h} - 1) B_{2h} \right] - (2n+1) \quad (\text{B.2.2.02})$$

La (B.2.2.02) è perfettamente identica alla (A.2.6.05)

B.2.3 – Ponendo nella (B.2.2.01), $x = 1/4$, otteniamo:

$$\int_0^{\infty} \frac{u^{2n} \text{Cosh}(u/4)}{\text{Cosh}(u/2)} du = \left[\frac{\pi}{\text{Cos}(\pi x)} \right]_{x=1/4}^{(2n)} = 2\pi \left[\sum_{k \geq 0} (-1)^k e^{-i\pi(1+2k)} \right]_{x=1/4}^{(2n)}$$

Operando, abbiamo:

$$\int_0^{\infty} \frac{u^{2n} \text{Cosh}(u/4)}{\text{Cosh}(u/2)} du = \int_0^{\infty} u^{2n} (e^{u/4} + e^{-u/4}) e^{-u/2} \sum_{k \geq 0} (-1)^k e^{-uk} du =$$

$$= \sum_{k \geq 0} (-1)^k \int_0^{\infty} u^{2n} (e^{u/4} + e^{-u/4}) e^{-u/2} e^{-uk} du = \sum_{k \geq 0} (-1)^k \int_0^{\infty} u^{2n} (e^{-u/4} + e^{-3u/4}) e^{-uk} du =$$

$$= (2n)! \sum_{k \geq 0} (-1)^k \left[\frac{1}{\left(\frac{1}{4} + k\right)^{2n+1}} + \frac{1}{\left(\frac{3}{4} + k\right)^{2n+1}} \right] = (2n)! 4^{2n+1} \sum_{k \geq 0} (-1)^k \left[\frac{1}{(1+4k)^{2n+1}} + \frac{1}{(3+4k)^{2n+1}} \right];$$

$$2\pi \left[\sum_{k \geq 0} (-1)^k e^{-i\pi(1+2k)} \right]_{x=1/4}^{(2n)} = 2\pi \sum_{k \geq 0} (-1)^k (-i\pi)^{2n} (1+2k)^{2n} e^{-i\pi/4} (-i)^k =$$

$$= 2\pi^{2n+1} (-1)^n e^{-i\pi/4} \sum_{k \geq 0} (i)^k (1+2k)^{2n};$$

$$\text{ora, } e^{-i\pi/4} \sum_{k \geq 0} (i)^k (1+2k)^{2n} = \frac{\sqrt{2}}{2} (1-i) [1 + \sum_{k \geq 1} (-1)^k (1+4k)^{2n} - i \sum_{k \geq 1} (-1)^k (4k-1)^{2n}];$$

prendendo le parti reali della relazione precedente, abbiamo:

$$\begin{aligned} \text{Re}[e^{-i\pi/4} \sum_{k \geq 0} (i)^k (1+2k)^{2n}] &= \frac{\sqrt{2}}{2} [1 + \sum_{k \geq 1} (-1)^k (1+4k)^{2n} - \sum_{k \geq 1} (-1)^k (1-4k)^{2n}] = \\ &= \frac{\sqrt{2}}{2} [1 + \sum_{k \geq 1} (-1)^k \sum_{h=0}^{2n} \binom{2n}{h} (4k)^h - \sum_{k \geq 1} (-1)^k \sum_{h=0}^{2n} \binom{2n}{h} (-4k)^h] = \\ &= \frac{\sqrt{2}}{2} [1 + \sum_{k \geq 1} (-1)^k \sum_{h=1}^n \binom{2n}{2h-1} (4k)^{2h-1} - \sum_{k \geq 1} (-1)^k \sum_{h=0}^n \binom{2n}{2h-1} (-4k)^{2h-1}] = \\ &= \frac{\sqrt{2}}{2} [1 + 2 \sum_{h=1}^n \binom{2n}{2h-1} 4^{2h-1} \sum_{k \geq 1} (-1)^k k^{2h-1}] = \\ &= \frac{\sqrt{2}}{2} [1 - 2 \sum_{h=1}^n \binom{2n}{2h-1} 4^{2h-1} (2^{2h} - 1) \frac{B_{2h}}{2h}] = \frac{\sqrt{2}}{2} [1 - \frac{1}{2(2n+1)} \sum_{h=1}^n \binom{2n+1}{2h} 4^{2h} (2^{2h} - 1) B_{2h}]; \end{aligned}$$

$$(2n)! 4^{2n+1} \sum_{k \geq 0} (-1)^k [\frac{1}{(1+4k)^{2n+1}} + \frac{1}{(3+4k)^{2n+1}}] = 2\pi^{2n+1} (-1)^n e^{-i\pi/4} \sum_{k \geq 0} (i)^k (1+2k)^{2n} =$$

$$= \pi^{2n+1} (-1)^{n-1} \sqrt{2} [\frac{1}{2(2n+1)} \sum_{h=1}^n \binom{2n+1}{2h} 4^{2h} (2^{2h} - 1) B_{2h} - 1], \text{ da cui:}$$

$$\begin{aligned} \sum_{k \geq 0} (-1)^k [\frac{1}{(1+4k)^{2n+1}} + \frac{1}{(3+4k)^{2n+1}}] &= \\ &= \frac{(-1)^{n-1} \pi^{2n+1} \sqrt{2}}{4^{2n+1} 2(2n+1)!} [\sum_{h=1}^n \binom{2n+1}{2h} 4^{2h} (2^{2h} - 1) B_{2h} - 2(2n+1)] \end{aligned} \quad (\text{B.2.3.01})$$

La relazione (B.2.3.01) è stata verificata con un programma di matematica.

B.2.4 Riconsideriamo la (B.2.2.01)

$$\int_0^{\infty} \frac{u^{2n} \text{Cosh}(ux)}{\text{Cosh}(u/2)} du = [\frac{\pi}{\text{Cos}(\pi x)}]^{(2n)} = 2\pi [\sum_{k \geq 0} (-1)^k e^{-i\pi x(1+2k)}]^{(2n)} \quad (\text{B.2.4.01})$$

Operando, abbiamo:

$$\begin{aligned}
\int_0^\infty \frac{u^{2n} \text{Cosh}(ux)}{\text{Cosh}(u/2)} du &= \int_0^\infty u^{2n} (e^{ux} + e^{-ux}) \sum_{k \geq 0} (-1)^k e^{-u/2} e^{-uk} du = \\
&= (2n)! \sum_{k \geq 0} (-1)^k \left[\frac{1}{\left(\frac{1}{2} - x + k\right)^{2n+1}} + \frac{1}{\left(\frac{1}{2} + x + k\right)^{2n+1}} \right]; \\
2\pi \left[\sum_{k \geq 0} (-1)^k e^{-i\pi x(1+2k)} \right]^{(2n)} &= 2\pi \left[\sum_{k \geq 0} (-1)^k (-i\pi)^{2n} (1+2k)^{2n} e^{-i\pi x(1+2k)} \right] = \\
&= 2\pi^{2n+1} (-1)^n \left[\sum_{k \geq 0} (-1)^k (1+2k)^{2n} \sum_{h \geq 0} \frac{(-i\pi x)^h (1+2k)^h}{h!} \right]
\end{aligned}$$

Per x reale, la precedente relazione è reale solamente quando h è pari, e quindi:

$$2\pi \left[\sum_{k \geq 0} (-1)^k e^{-i\pi x(1+2k)} \right]^{(2n)} = 2\pi^{2n+1} (-1)^n \sum_{h \geq 0} \frac{(-i\pi x)^{2h}}{(2h)!} \sum_{k \geq 0} (-1)^k (1+2k)^{2n+2h} =$$

Applicando la (A.2.5.01), abbiamo:

$$\begin{aligned}
\sum_{k \geq 0} (-1)^k (1+2k)^{2n+2h} &= \frac{1}{2} - \frac{1}{2(2n+2h)} \sum_{j=1}^{n+h} \binom{2n+2h+1}{2j} 2^{2j} (2^{2j} - 1) B_{2j}; \text{ pertanto:} \\
(2n)! \sum_{k \geq 0} (-1)^k \left[\frac{1}{\left(\frac{1}{2} - x + k\right)^{2n+1}} + \frac{1}{\left(\frac{1}{2} + x + k\right)^{2n+1}} \right] &= \\
= \pi^{2n+1} (-1)^n \sum_{h \geq 0} \frac{(\pi x)^{2h} (-1)^h}{(2h)!} \left[1 - \frac{1}{2n+2h} \sum_{j=1}^{n+h} \binom{2n+2h+1}{2j} 2^{2j} (2^{2j} - 1) B_{2j} \right], & \quad (\text{B.2.4.02})
\end{aligned}$$

Per $x = 0$, ritroviamo la (B.2.2.02);

per $x = 1/4$, ritroviamo la (B.2.3.01).

Inoltre, operando sulla (B.2.3.01), otteniamo:

$$\begin{aligned}
\left[\frac{\pi}{\text{Cos}(\pi x)} \right]^{(2n)} &= \left[\frac{\pi}{\text{Cos}(\pi x)} \right]^{(2n-1+1)} = \pi \left[\frac{\pi \text{Sin}(\pi x)}{(\text{Cos}(\pi x))^2} \right]^{(2n-1)} = \\
&= \pi \left[\text{Sin}(\pi x) D \text{Tan}(\pi x) \right]^{(2n-1)} = \pi \sum_{K=0}^{2n-1} \binom{2n-1}{K} \left[\text{Sin}(\pi x) \right]^{(2n-1-K)} \left[\text{Tan}(\pi x) \right]^{(K+1)} = \\
&= \pi \sum_{K=0}^{n-1} \binom{2n-1}{2K} \left[\text{Sin}(\pi x) \right]^{(2n-1-2K)} \left[\text{Tan}(\pi x) \right]^{(2K+1)} + \pi \sum_{K=1}^n \binom{2n-1}{2K-1} \left[\text{Sin}(\pi x) \right]^{(2n-2K)} \left[\text{Tan}(\pi x) \right]^{(2K)};
\end{aligned}$$

Ricordando le relazioni (B.1.3.03) e (B.1.3.04), abbiamo:

$$\left[\text{Sin}(\pi x) \right]^{(2n-1-2k)} = \pi^{2n-1-2k} (-1)^{n-k-1} \text{Cos}(\pi x) \quad (\text{B.2.4.03})$$

$$\left[\text{Sin}(\pi x) \right]^{(2n-2k)} = \pi^{2n-2k} (-1)^{n-k} \text{Sin}(\pi x) \quad (\text{B.2.4.04})$$

Applicando la (A.1.1.01) e la (A.2.1.01), ricaviamo:

$$\begin{aligned}
\left[\frac{\pi}{\text{Cos}(\pi x)} \right]^{(2n)} &= \pi^{2n+1} \sum_{K=0}^{n-1} \binom{2n-1}{2K} (-1)^{n-k-1} \text{Cos}(\pi x) \sum_{h=0}^{K+1} a_h \left[\text{Tan}(\pi x) \right]^{2h} + \\
&\quad + \pi^{2n+1} \sum_{K=1}^n \binom{2n-1}{2K-1} (-1)^{n-k} \text{Sin}(\pi x) \sum_{h=0}^K b_h \left[\text{Tan}(\pi x) \right]^{2h+1} \quad (\text{B.2.4.05})
\end{aligned}$$

I coefficienti a_h, b_h sono forniti, rispettivamente, dalle formule (A.1.3.05) e (A.2.3.03).

$$\text{Pertanto: } (2n)! \sum_{k \geq 0} (-1)^k \left[\frac{1}{\left(\frac{1}{2} - x + k\right)^{2n+1}} + \frac{1}{\left(\frac{1}{2} + x + k\right)^{2n+1}} \right] =$$

$$\begin{aligned}
&= \pi^{2n+1} (-1)^{n-1} \left\{ \sum_{K=0}^{n-1} \binom{2n-1}{2k} (-1)^k \text{Cos}(\pi x) \sum_{h=0}^{k+1} a_h \text{Tan}(\pi x) \right\}^{2h} + \\
&\quad - \sum_{K=1}^n \binom{2n-1}{2k-1} (-1)^k \text{Sin}(\pi x) \sum_{h=0}^k b_h [\text{Tan}(\pi x)]^{2h+1} \}; \quad |x| < \frac{1}{2}, \quad (\text{B.2.4.06})
\end{aligned}$$

per $x = 0$, abbiamo: $(2n)! 2^{2n+2} \sum_{k \geq 0} \frac{(-1)^k}{(1+2k)^{2n+1}} = \pi^{2n+1} (-1)^{n-1} \left\{ \sum_{K=0}^{n-1} \binom{2n-1}{2k} (-1)^k a_0 = \right.$

$$= \pi^{2n+1} (-1)^{n-1} \sum_{K=0}^{n-1} \binom{2n-1}{2k} (-1)^k 2^{2k+2} (2^{2k+2} - 1) \frac{|B_{2k+2}|}{2k+2}; \quad (\text{B.2.4.07})$$

Per $x = 1/4$, otteniamo:

$$\begin{aligned}
(2n)! 4^{2n+1} \sum_{k \geq 0} (-1)^k \left[\frac{1}{(1+4k)^{2n+1}} + \frac{1}{(3+4k)^{2n+1}} \right] &= \pi^{2n+1} (-1)^{n-1} \frac{\sqrt{2}}{2} \left\{ \sum_{K=0}^{n-1} \binom{2n-1}{2k} (-1)^k \sum_{h=0}^{k+1} a_h + \right. \\
&\quad \left. - \sum_{K=1}^n \binom{2n-1}{2k-1} (-1)^k \sum_{h=0}^k b_h \right\}
\end{aligned}$$

C.0.0- Utilizzando la nota relazione

$$\int_0^\infty \frac{t^{z-1} dt}{1+t} = \frac{\pi}{\text{Sin} \pi z}, \quad 0 < z < 1,$$

e ponendo $z = 1/n$, con $n > 1$, otteniamo:

$$\begin{aligned}
\int_0^\infty \frac{t^{\frac{1}{n}-1} dt}{1+t} &= \frac{\pi}{\text{Sin} \frac{\pi}{n}} = (t = u^n) = \int_0^\infty \frac{u^{n(\frac{1}{n}-1)} n u^{n-1} du}{1+u^n} = n \int_0^\infty \frac{du}{1+u^n} = \\
&= n \left[\int_0^1 \frac{du}{1+u^n} + \int_0^1 \frac{u^{n-2} du}{1+u^n} \right] = n \int_0^1 \frac{1+u^{n-2}}{1+u^n} du.
\end{aligned}$$

Ora, $\frac{n(1+u^{n-2})}{1+u^n} = n \sum_{k=0}^{n-1} \frac{1+u_k^{n-2}}{u-u_k} \frac{1}{n u_k^{n-1}} = \sum_{k=0}^{n-1} \frac{1+u_k^{n-2}}{u-u_k} \frac{u_k}{u_k^n}$,

dove $u_k^n = -1$, $u_k = e^{i\pi(2k+1)/n}$, $k = 0, 1, 2, 3, \dots, (n-1)$; quindi:

$$\frac{n(1+u^{n-2})}{1+u^n} = - \sum_{k=0}^{n-1} \left(u_k - \frac{1}{u_k} \right) \frac{1}{u-u_k}; \quad \text{pertanto:}$$

$$\int_0^\infty \frac{t^{\frac{1}{n}-1} dt}{1+t} = \frac{\pi}{\text{Sin} \frac{\pi}{n}} = - \sum_{k=0}^{n-1} \left(u_k - \frac{1}{u_k} \right) [\ln(u-u_k)]_0^1 = - \sum_{k=0}^{n-1} \left(u_k - \frac{1}{u_k} \right) \left[\ln\left(1 - \frac{1}{u_k}\right) \right] =$$

$$= - \sum_{k=0}^{n-1} (e^{i\pi(2k+1)/n} - e^{-i\pi(2k+1)/n}) \ln(1 - e^{-i\pi(2k+1)/n}); \quad (\text{C.0.0.01})$$

$$\ln[1 - e^{-i\pi(2k+1)/n}] = \ln\{1 - \text{Cos}[\pi(2k+1)/n] + i \text{Sin}[\pi(2k+1)/n]\} =$$

$$= \frac{1}{2} \ln\{(1 - \text{Cos}[\pi(2k+1)/n])^2 + (\text{Sin}[\pi(2k+1)/n])^2\} + i \text{ArcTan} \frac{\text{Sin}[\pi(2k+1)/n]}{1 - \text{Cos}[\pi(2k+1)/n]};$$

$$iArcTan \frac{Sin[\pi(2k+1)/n]}{1-Cos[\pi(2k+1)/n]} = iArcTan \frac{Cos[\pi(2k+1)/(2n)]}{Sin[\pi(2k+1)/(2n)]} =$$

$$= iArcTan \frac{Sin\{(\pi/2)[1-(2k+1)/n]\}}{Cos\{(\pi/2)[1-(2k+1)/n]\}} = i \frac{\pi}{2} \left(1 - \frac{2k+1}{n}\right).$$

Sostituendo nella (C.0.0.01), ed uguagliando le parti reali, ricaviamo:

$$\int_0^{\infty} \frac{t^{\frac{1}{n}-1}}{1+t} dt = \frac{\pi}{Sin \frac{\pi}{n}} = \sum_{k=0}^{n-1} 2Sin[\pi(2k+1)/n] \left\{ \frac{\pi}{2} \left(1 - \frac{2k+1}{n}\right) \right\} =$$

$$= \pi \sum_{k=0}^{n-1} \left(\frac{n-2k-1}{n} \right) Sin[\pi(2k+1)/n], \text{ cioè:}$$

$$\sum_{k=0}^{n-1} (n-2k-1) Sin[\pi(2k+1)/n] Sin \frac{\pi}{n} = n$$

Ricordando che : $Cos(a+b) - Cos(a-b) = -2Sin(a)Sin(b)$, otteniamo:

$$\sum_{k=0}^{n-1} (n-2k-1) \left(-\frac{1}{2}\right) \left[Cos \frac{2\pi(k+1)}{n} - Cos \frac{2\pi k}{n} \right] = n;$$

sostituendo, nella precedente relazione, k a $k+1$, troviamo:

$$\sum_{k=1}^n (n-2k+1) \left[Cos \frac{2\pi(k-1)}{n} - Cos \frac{2\pi k}{n} \right] = 2n; \quad (C.0.0.02)$$

Inoltre, osserviamo che:

$$\sum_{k=1}^n \left[Cos \frac{2\pi(k-1)}{n} - Cos \frac{2\pi k}{n} \right] = 0, \text{ e quindi dalla (C.0.0.02), ricaviamo:}$$

$$\sum_{k=1}^n (-2k) \left[Cos \frac{2\pi(k-1)}{n} - Cos \frac{2\pi k}{n} \right] = 2n, \text{ cioè:}$$

$$\sum_{k=1}^n k \left[Cos \frac{2\pi k}{n} - Cos \frac{2\pi(k-1)}{n} \right] = n \quad (C.0.0.03)$$

La relazione (C.0.0.03) è stata verificata con un programma di matematica.

Riteniamo utile riportare in un unico schema le interessanti formule trovate.

$$1) P(x) = Tan(x), \quad [Tan(x)]^{(2n-1)} = \sum_{h=0}^n a_h [Tan(x)]^{2h}$$

$$a_0 = 2^{2n} (-1)^{n-1} (2^{2n} - 1) \frac{B_{2n}}{2n} = 2^{2n} (2^{2n} - 1) \frac{|B_{2n}|}{2n}, \quad (A.1.1.06)$$

$$2) \sum_{k \geq 0} \frac{1}{(1+2k)^{2n}} = \frac{(2^{2n} - 1)\pi^{2n}}{2(2n)!} |B_{2n}| \quad (A.1.1.10)$$

$$3) \sum_{h=0}^n a_h = 2^{4n-1} (2^{2n} - 1) \frac{|B_{2n}|}{2n}, \quad (A.1.2.01)$$

$$4) \quad \left(\sum_{h=0}^n a_h \right) / a_0 = 2^{2n-1} \quad (\text{A.1.2.02})$$

$$5) \quad a_h = \frac{2^{2n}}{(2h)!} (-1)^{n-1} (-1)^h \left[\sum_{u=0}^h s(2h, 2u) (2^{2n+2u} - 1) \frac{B_{2n+2u}}{2n+2u} + \right. \\ \left. + 2h \sum_{u=1}^h s(2h-1, 2u-1) (2^{2n+2u} - 1) \frac{B_{2n+2u}}{2n+2u} + \right. \\ \left. + \sum_{j=2}^{2h} \binom{2h}{j} \sum_{u=0}^{2h-1} s(2h-1, u) \sum_{p=1}^{\lfloor (u+1)/2 \rfloor} \binom{u}{2p-1} (j-1)^{u-2p+1} (2^{2n+2p} - 1) \frac{B_{2n+2p}}{2n+2p} \right]. \quad (\text{A.1.3.05})$$

$$6) a_n = -\frac{2^{2n}}{(2n)!} \left[\sum_{u=0}^n s(2n, 2u) (2^{2n+2u} - 1) \frac{B_{2n+2u}}{2n+2u} + \right. \\ \left. + 2n \sum_{u=1}^n s(2n-1, 2u-1) (2^{2n+2u} - 1) \frac{B_{2n+2u}}{2n+2u} + \right. \\ \left. + \sum_{j=2}^{2n} \binom{2n}{j} \sum_{u=0}^{2n-1} s(2n-1, u) \sum_{p=1}^{\lfloor u/2 \rfloor} \binom{u}{2p-1} (j-1)^{u-2p+1} (2^{2n+2p} - 1) \frac{B_{2n+2p}}{2n+2p} \right] = (2n-1)!, \quad (\text{A.1.3.06})$$

$$7) \sum_{k \geq 0} \left[\frac{1}{\left(\frac{1}{2} - x + k\right)^{2n}} + \frac{1}{\left(\frac{1}{2} + x + k\right)^{2n}} \right] = \frac{2^{2n}}{(2n-1)!} \pi^{2n} \sum_{h \geq 0} \frac{(2\pi x)^{2h}}{(2h)!} \frac{(2^{2n+2h} - 1) |B_{2n+2h}|}{2n+2h} = \\ = \frac{\pi^{2n}}{(2n-1)!} \sum_{h=0}^n a_h [\text{Tan}(\pi x)]^{2h} \quad (\text{A.1.5.06})$$

$$(8) \quad [\text{Tan}(x)]^{(2n)} = \sum_{h=0}^n b_h [\text{Tan}(x)]^{2h+1}$$

$$b_0 = \frac{2^{2n+1}}{n+1} (2^{2n+2} - 1) |B_{2n+2}| \quad (\text{A.2.1.03})$$

$$9) \sum_{k \geq 1} (-1)^k (2k-1)^{2n} = -\frac{1}{2} + \frac{1}{2(2n+1)} \sum_{h \geq 1} \binom{2n+1}{2h} (-2)^{2h} (2^{2h} - 1) B_{2h}, \quad (\text{A.2.2.03})$$

$$10) \sum_{h=0}^n b_h = 2^{2n} (-1)^{n-1} \left[\frac{1}{2n+1} \sum_{h=1}^n \binom{2n+1}{2h} 2^{2h} (2^{2h} - 1) B_{2h} - 1 \right] \quad (\text{A.2.2.04})$$

$$11) b_h = \frac{2^{2n+1}}{(2h+1)!} (-1)^n (-1)^h \left[\sum_{u=0}^h s(2h+1, 2u+1) (2^{2n+2u+2} - 1) \frac{B_{2n+2u+2}}{2n+2u+2} + \right. \\ \left. + (2h+1) \sum_{u=0}^h s(2h, 2u) (2^{2n+2u+2} - 1) \frac{B_{2n+2u+2}}{2n+2u+2} + \right. \\ \left. + \sum_{j=2}^{2h+1} \binom{2h+1}{j} \sum_{u=0}^{2h} s(2h, u) \sum_{p=0}^{\lfloor u/2 \rfloor} \binom{u}{2p} (j-1)^{u-2p} (2^{2n+2p+2} - 1) \frac{B_{2n+2p+2}}{2n+2p+2} \right] \quad (\text{A.2.3.03})$$

$$\begin{aligned}
12) \quad b_n &= \frac{2^{2n+1}}{(2n+1)!} \left[\sum_{u=0}^n s(2n+1, 2u+1) (2^{2n+2u+2} - 1) \frac{B_{2n+2u+2}}{2n+2u+2} + \right. \\
&+ (2n+1) \sum_{u=0}^n s(2n, 2u) (2^{2n+2u+2} - 1) \frac{B_{2n+2u+2}}{2n+2u+2} \\
&+ \left. \sum_{j=2}^{2n+1} \binom{2n+1}{j} \sum_{u=0}^{2n} s(2n, u) \sum_{p=0}^{\lfloor u/2 \rfloor} \binom{u}{2p} (j-1)^{u-2p} (2^{2n+2p+2} - 1) \frac{B_{2n+2p+2}}{2n+2p+2} \right] = (\mathbf{2n})! \quad (\text{A.2.3.04})
\end{aligned}$$

$$13) \quad \frac{2}{2n+1} \sum_{h=1}^n \binom{2n+1}{2h} (2^{2h} - 1) B_{2h} = 1 \quad (\text{A.2.4.02})$$

$$14) \quad \sum_{k \geq 0} (-1)^k (1+2k)^{2n} = \frac{1}{2} - \frac{1}{2(2n+1)} \sum_{h \geq 1} \binom{2n+1}{2h} 2^{2h} (2^{2h} - 1) B_{2h}; \quad (\text{A.2.5.01})$$

$$15) \quad \sum_{k \geq 0} \frac{(-1)^k}{(1+2k)^{2n+1}} = \frac{\pi^{2n+1} (-1)^{n-1}}{2^{2n+2} (2n+1)!} \left[\sum_{h=1}^n \binom{2n+1}{2h} 2^{2h} (2^{2h} - 1) B_{2h} - (2n+1) \right] \quad (\text{A.2.5.02})$$

$$\begin{aligned}
16) \quad \sum_{k \geq 0} \left[\frac{1}{(1+4k)^{2n+1}} - \frac{1}{(3+4k)^{2n+1}} \right] &= \frac{1}{(2n)! 4^{2n+1}} \sum_{h=0}^n b_h = \\
&= \frac{(2\pi)^{2n+1}}{(2n)! 4^{2n+1}} \sum_{h \geq 1} \frac{(\pi/2)^{2h-1} (2^{2n+2h} - 1) |B_{2n+2h}|}{(2h-1)! 2n+2h} \quad (\text{A.2.6.05})
\end{aligned}$$

$$17) \quad \sum_{k \geq 0} (-1)^k (1+2k)^{2n-1} = 0 \quad (\text{B.1.1.02})$$

$$18) \quad \frac{1}{2n} \sum_{h=1}^n \binom{2n}{2h} 2^{2h} (2^{2h} - 1) B_{2h} = 1 \quad (\text{B.1.1.03})$$

$$\begin{aligned}
19) \quad \sum_{k \geq 0} (-1)^k \left[\frac{1}{(1+4k)^{2n}} - \frac{1}{(3+4k)^{2n}} \right] &= \\
&= \pi^{2n} (-1)^{n-1} \sqrt{2} \frac{4^{-2n}}{2(2n)!} \left[\sum_{h=1}^n \binom{2n}{2h} 4^{2h} (2^{2h} - 1) B_{2h} - 4n \right]. \quad (\text{B.1.2.01})
\end{aligned}$$

$$\begin{aligned}
20) \quad (2n-1)! \sum_{k \geq 0} (-1)^k \left[\frac{1}{\left(\frac{1}{2} - x + k\right)^{2n}} - \frac{1}{\left(\frac{1}{2} + x + k\right)^{2n}} \right] &= \\
&= \pi^{2n} (-1)^{n-1} \text{Sin}(\pi x) \left\{ \sum_{k=0}^{n-1} \binom{2n-2}{2k} (-1)^k \sum_{h=0}^{k+1} a_h \text{Tan}(\pi x) \right\}^{2h} + \\
&+ \sum_{k=1}^{n-1} \binom{2n-2}{2k-1} (-1)^k \sum_{h=0}^k b_h [\text{Tan}(\pi x)]^{2h} \quad (\text{B.1.3.05})
\end{aligned}$$

$$\begin{aligned}
21) \sum_{k \geq 0} (-1)^k \left[\frac{1}{(1+4k)^{2n+1}} + \frac{1}{(3+4k)^{2n+1}} \right] &= \\
&= \frac{(-1)^{n-1} \pi^{2n+1} \sqrt{2}}{4^{2n+1} 2(2n+1)!} \left[\sum_{h=1}^n \binom{2n+1}{2h} 4^{2h} (2^{2h} - 1) B_{2h} - 2(2n+1) \right] \quad (B.2.3.01)
\end{aligned}$$

$$\begin{aligned}
22) \quad (2n)! \sum_{k \geq 0} (-1)^k \left[\frac{1}{\left(\frac{1}{2} - x + k\right)^{2n+1}} + \frac{1}{\left(\frac{1}{2} + x + k\right)^{2n+1}} \right] &= \\
&= \pi^{2n+1} (-1)^n \sum_{h \geq 0} \frac{(\pi x)^{2h} (-1)^h}{(2h)!} \left[1 - \frac{1}{2n+2h} \sum_{j=1}^{n+h} \binom{2n+2h+1}{2j} \right] 2^{2j} (2^{2j} - 1) B_{2j}, \quad (B.2.4.02)
\end{aligned}$$

$$\begin{aligned}
23) \quad (2n)! \sum_{k \geq 0} (-1)^k \left[\frac{1}{\left(\frac{1}{2} - x + k\right)^{2n+1}} + \frac{1}{\left(\frac{1}{2} + x + k\right)^{2n+1}} \right] &= \\
&= \pi^{2n+1} (-1)^{n-1} \left\{ \sum_{K=0}^{n-1} \binom{2n-1}{2K} (-1)^K \text{Cos}(\pi x) \sum_{h=0}^{K+1} a_h \text{Tan}(\pi x) \right\}^{2h} + \\
&\quad - \sum_{K=1}^n \binom{2n-1}{2K-1} (-1)^K \text{Sin}(\pi x) \sum_{h=0}^K b_h [\text{Tan}(\pi x)]^{2h+1}; \quad |x| < \frac{1}{2}, \quad (B.2.4.06)
\end{aligned}$$

$$24) \quad \sum_{k=1}^n k \left[\text{Cos} \frac{2\pi k}{n} - \text{Cos} \frac{2\pi(k-1)}{n} \right] = n \quad (C.0.0.03)$$

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Dott. Ing. Pasquale Cutolo

Riferimenti

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